CPE300: Digital System Architecture and Design

Fall 2011

MW 17:30-18:45 CBC C316

Arithmetic Unit

10102011

http://www.egr.unlv.edu/~b1morris/cpe300/

Chapter 6

- Number Systems and Radix Conversion
- Fixed-Point Arithmetic
- Seminumeric Aspects of ALU Design
- Floating-Point Arithmetic

Outline

- Number Systems
- Fixed Point Arithmetic

Digital Number Systems

- Expanded generalization of lecture 07 topics

- Number systems have a base (radix) b
- Positional notation of an m digit base b number
  - $x = x_{m-1}x_{m-2} \ldots x_1 x_0$
  - $\text{Value}(x) = \sum_{i=0}^{m-1} x_i b^i$

Range of Representation

- Largest number has all digits equal to largest possible base b digit, $(b-1)$
- Max value in closed form for unsigned m digit base b number
  - $x_{\max} = \sum_{i=0}^{m-1}(b-1)b^i$
  - $x_{\max} = (b-1)\sum_{i=0}^{m-1} b^i = (b-1)\left(\frac{b^m-1}{b-1}\right)$
  - $x_{\max} = b^m - 1$

- Sum of geometric series
  - $\sum_{i=0}^{m-1} b^i = \left(\frac{b^m-1}{b-1}\right)$

Radix Conversion

- Conversion between different number systems involves computation
  - Base of calculation is c (10 typical for us humans)
  - Other base is b
- Calculation based on division
  - For integers a and d, exist integers q and r such that
  - $a = q \cdot d + r$
    - $0 \leq r \leq b - 1$
- Notation:
  - $q = \lfloor a/d \rfloor$
  - $r = a \bmod b$ (mod is remainder)

Digit Symbol Correspondence Between Bases

- Each base (b or c) has different symbols to represent digits
- Lookup table given for correspondence between symbols
  - Provides mapping between base b and base c symbols
  - May be more than one digit required to represent a larger base symbol

Base 12	0	1	2	3	4	5	6	7	8	9	A	B
Base 3	0	1	2	10	11	12	20	21	22	100	101	102

Base Conversion 1

- Convert base b integer to calculator base c

1. Start with base b
   - $x = x_{m-1}x_{m-2} \dots x_1x_0$
2. Set $x = 0$ in base c
3. Left to right, get next symbol x_i
4. Lookup base c number D_i for symbol x_i
5. Calculate in base c
   - $x = x \cdot b + Di$
6. Repeat step 3 until no more digits

- Example:
- Convert `0x3AF` to base 10
  - $x = 0$
  - $x = 16 \cdot 0 + 3 = 3$
  - $x = 16 \cdot 3 + 10\ (= A) = 58$
  - $x = 16 \cdot 58 + 15\ (= F) = 943$
- `0x3AF` = 943_{10}

Base Conversion 2

- Convert calculator base c integer to base b

1. Start with base c integer
   - $x = x_{m-1}x_{m-2} \dots x_1x_0$
2. Initialize
   - $i = 0$
   - $v = x$
   - Produce digits right to left
3. Set
   - $D_i = v \bmod b$
   - $v = \lfloor v/b \rfloor$
   - Lookup D_i to get x_i
4. Set
   - $i = i + 1$
   - Repeat step 3 if $v \neq 0$

- Example:
- Convert 3587_{10} to base 12
  - $\frac{3587}{12} = 298\ (rem = 11) \Rightarrow x_0 = B$
  - $\frac{298}{12} = 24\ (rem = 10) \Rightarrow x_1 = A$
  - $\frac{24}{12} = 2\ (rem = 0) \Rightarrow x_2 = 0$
  - $\frac{2}{12} = 0\ (rem = 2) \Rightarrow x_3 = 2$
- $3587 = 20AB_{12}$

Fractions and Fixed Point Numbers

- Base b fraction
  - $f = .f_{-1}\, f_{-2} \ldots\, f_{-m}$
  - Value is integer $f_{-1}\, f_{-2} \ldots\, f_{-m}$ divided by b^m
- Mixed fixed point number
  - $x_{n-1}x_{n-2} \ldots x_1x_0 . x_{-1}x_{-2} \ldots x_{-m}$
  - Value of n+m digit integer
    - $x_{n-1}x_{n-2} \ldots x_1x_0x_{-1}x_{-2} \ldots x_{-m}$
  - Divided by b^m
- Moving radix point one place left divides by b
  - Right shift for fixed radix point position
- Moving radix point one place right multiplies by b
  - Left Shift for fixed radix point position

Converting Fractions to Calculator Base

- Can use integer conversion and divide result by b^m
- Alternative algorithm

1. Let base b number be
   - $f = .f_{-1}f_{-2} \dots f_{-m}$
2. Initialize
   - $f = 0.0$
   - $i = -m$
3. Find base c equivalent of D of digit f_i
4. Update
   - $f = \frac{f+D}{b}$
   - $i = i + 1$
5. If $i = 0$, result is f; otherwise repeat step 3

- Example
- Convert 0.413_8 to base 10
  - $f = \frac{0+3}{8} = 0.375$
  - $f = \frac{(0.375+1)}{8} = 0.171875$
  - $f = \frac{0.171875+4}{8} = 0.521484375$
- Notice: there will be precision errors due to numerical round-off
  - Only a fixed number of digits can be retained

Converting Fractions to Base b

1. Start with fraction f in base c
   - $f = .f_{-1}f_{-2} \dots f_{-m}$
2. Initialize
   - $v = f$
   - $i = 1$
3. Set
   - $D_{-i} = \lfloor b \cdot v \rfloor$
   - $v = b \cdot v\text{-}D_{-i}$
   - Get base b digit f_{-i} for D_{-i} with table
4. Increment
   - $i = i + 1$
   - Repeat Step 3 until
     - $v = 0$
     - Enough digits generated

- Example
- Convert 0.31_{10} to base 8
  - $0.31 \times 8 = 2.48 \Rightarrow f_{-1} = 2$
  - $0.48 \times 8 = 3.84 \Rightarrow f_{-2} = 3$
  - $0.84 \times 8 = 6.72 \Rightarrow f_{-3} = 6$
- $f = 0.236_8$
- Notice:
  - Since $8^3 > 10^2$, 0.236_8 has more accuracy than 0.31_{10}

Digit Grouping for Related Bases

- Base b = c^k
- Can convert between bases by replacing single digit symbol in base b with corresponding digits in base c

- (Our favorite method to change base e.g. binary to hex)
- Examples
  - $102130_4 = 10\ 21\ 30_4 = 0\text{x}49\text{C}$

Negative Numbers and Complements

- Two complement operations defined
- Two complement number systems
  - Represent both positive and negative numbers

- Given m digit base b number x
- Radix complement (b's complement)
  - $x^c = (b^m - x) \bmod b^m$
  - mod b^m only has effect for x=0
    - What is radix complement of x = 0?
- Diminished radix complement ((b-1)'s complement)
  - $\hat{x}_c = b^m - 1 - x$

Complement Number Systems

- Both positive and negative numbers represented in m digits
  - Range of m digit base b unsigned number:
    - $0 \leq x \leq b^m - 1$
- First half of range used for positive and second half for negative numbers
  - Complement of number range
    - Positive: 0 to $b^m/2$
    - Negative: $b^m/2$ to b^m-1
  - Radix complement has extra negative number for even b (think b=2)
  - Diminished radix complement has equal numbers of positive and negative representations

Utility of Complement System

- Sign-magnitude system requires extra +/- symbols in addition to digits
  - Binary has easy mapping
    - + := 0
    - - := 1
  - If b > 2 a whole digit for the 2 +/- symbols is wasteful
- Easy to do signed addition and subtraction using the complement number systems

Complement Representation of Negative Numbers

Radix Complement		Diminished Radix Complement	
Number	Representation	Number	Representation
0	0	0	0 or $b^m - 1$
$0 < x < b^m/2$	x	$0 < x < bm/2$	x
$-b^m/2 \leq x < 0$	$\lvert x\rvert^c = b^m - \lvert x\rvert$	$-bm/2 \leq x < 0$	$\widehat{\lvert x\rvert}^c = b^m - 1 - \lvert x\rvert$

- Radix complement has one more negative than positive for even base b
- Diminished radix complement has 2 zeros but same number of positive and negative values

Base 2 Complement Representations

8 Bit Radix (2's)Complement		8 bit Diminished Radix (1's) Complement	
Number	Representation	Number	Representation
0	0	0	0 or 255
$0 < x < 128$	x	$0 < x < 128$	x
$-128 \leq x < 0$	$256 - \lvert x \rvert$	$-127 \leq x < 0$	$256 - 1 - \lvert x \rvert$

- 1's complement 255 (or -0)
  - $255 = 1111\ 1111_2$
- 2's complement
  - $-128 = 1000\ 0000_2$ is valid
  - Negation gives overflow

Negation in Complement Systems

- Negative of any m digit value is also m digits
  - Exception: $-b^m/2$
- Negative of any number is obtained by applying the b's or (b-1)'s complement operation
- The complement operations are related
  - $x^c = (\hat{x}^c + 1) \bmod b^m$
  - Given one, easy to compute other

Digitwise Computation of Diminished Radix Complement

- $\hat{x}_c = bm - 1 - x$
- $\hat{x}_c = \sum_{i=0}^{m-1}(b-1)b^i - \sum_{i=0}^{m-1}(x_i)b^i$
- $\hat{x}_c = \sum_{i=0}^{m-1}(b-1-x_i)b^i$

- Diminished radix number is an m digit base b number
  - Each digit is obtained (as diminished complement) from corresponding digit in x

Base 5 Complements

Base 5 Digit	0	1	2	3	4
4's Comp.	4	3	2	1	0

- Examples
- 4's complement of 201341_5
  - 243103_5
- 5's complement of 201341_5
  - $243103_5 + 1 = 243104_5$
- 5's complement of 44444_5
  - $00000_5 + 1 = 00001_5$
- 5's complement of 00000_5
  - $(44444_5 + 1) \bmod 5^5 = 00000_5$

Complement Fractions

- m digit fraction is same as m digit integer divided by b^m,
  - The b^m in complement definitions corresponds to 1 for fractions
- Radix complement of $f = .f_{-1}f_{-2}\ldots f_{-m}$
  - $(1-x) \bmod 1$
    - Where mod 1 means discard integer
- The range of fractions is roughly -1/2 to +1/2
- This can be inconvenient for a base other than 2
- The b's comp. of a mixed number
  - $x = x_{m-1}x_{m-2}\ldots x_1x_0.x_{-1}x_{-2}\ldots x_{-n} = b^m - x$,
  - Both integer and fraction digits are subtracted

Scaling Complement Numbers

- Dividing by b corresponds to moving radix point one place left
  - Shift number one place right
- Multiplying by base b corresponds to moving radix point one place right (roughly)
  - Shift number one place left

- Issues:
  - What is new left digit on right shift?
  - When does left shift overflow?

Right Shift for Divide

- Positive number $x = x_{m-1}x_{m-2} \dots x_1x_0$
  - Zero fill: $x/b = 0x_{m-1}x_{m-2} \dots x_1$
- Negative number
  - (b-1) file: $x/b = (b-1)x_{m-1}x_{m-2} \dots x_1$

- Fill rule for even b
  - Zero fill when $x_{m-1} < b/2$
  - (b-1) fill when $x_{m-1} \geq b/2$

Left Shift for Multiply

- Overflow can occur (loss of information)
  - Positive numbers
    - Any digit other than 0 shifts off left end
    - After shift, left-most digit makes number look negative (digit $\geq b/2$ for even b)
  - Negative numbers
    - Any digit other than (b-1) shifts off left end
    - After shirt, left-most digit makes number look positive (digit $< b/2$ for even b)

Left Shift Examples

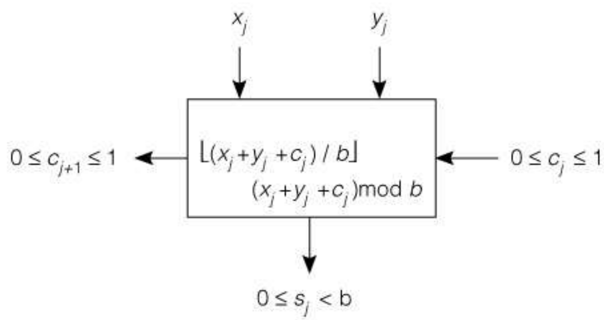
- Non-overflow cases:
  - $762_8 << 1 = 620_8$; $-14 * 8 = -112$
  - $031_8 << 1 = 310_8$; $-25 * 8 = -200$
- Overflow cases
  - $241_8 << 1 = 410_8$; $2 \neq 0$ off left
  - $041_8 << 1 = 410_8$; changes from + to –
  - $713_8 << 1 = 130_8$; changes from – to +
  - $662_8 << 1 = 620_8$; $2 \neq 7$ off left

Fixed Point Addition and Subtraction

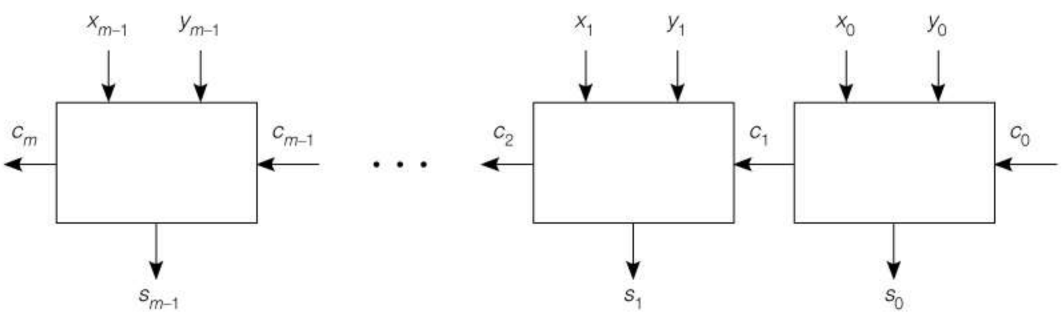
- When radix point is in the same position for both operands
  - Add/Sub acts as if numbers were integers
- Addition of signed numbers in radix complement system only needs an unsigned adder
  - Must design m digit base b unsigned adder
- Radix complement signed addition theorem
  - s = rep(x) + rep(y) = rep(x+y)
  - rep(x) := b's complement representation of x
  - Does not consider overflow

Unsigned Addition Hardware

- Perform operation on each digit of m digit base b number
- Each digit cell requires operands x_j and y_j as well as a carry in c_j
- Sum
  - $s_j = (x_j + y_j + c_j) \bmod b$
- Carry-out
  - $c_{j+1} = \lfloor (x_j + yj + cj)/b \rfloor$
  - All carries are less than equal to 1 regardless of b
- Works for any fixed radix point location (e.g. fractions)



Base *b* digit adder



An *m*-digit base *b* unsigned adder

Unsigned Addition Example

Op1		1	2	.	0	3_4	=	6.1875_{10}
Op2	+	1	3	.	2	1_4	=	7.5625_{10}
Carry	0	1	0		1			
Sum		3	1	.	3	0_4	=	13.75

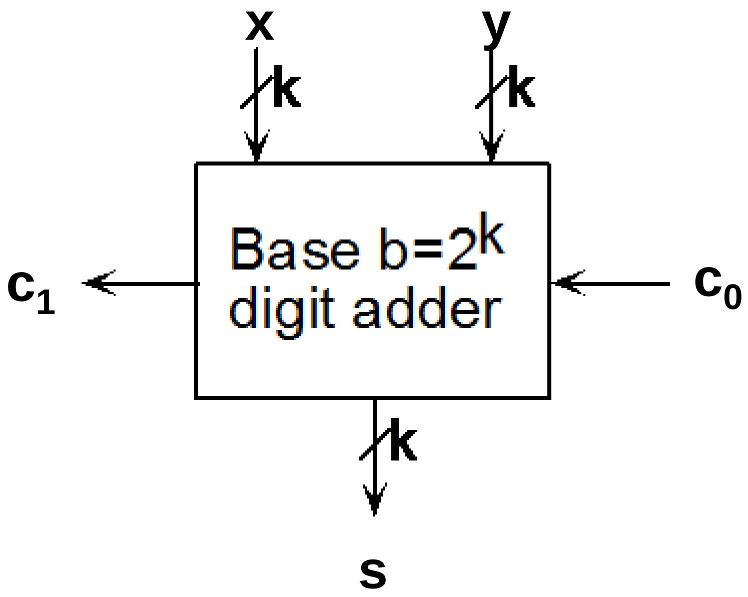
+	0	1	2	3
0	00	01	02	03
1	01	02	03	10
2	02	03	10	11
3	03	10	11	12

Base 4 addition table

- With fixed number of digits, overflow occurs on carry from leftmost digit
- Carries are 0 or 1 in all cases
- Addition is defined by a table of sum and carry for b^2 digit pairs

Adder Implementation Alternatives

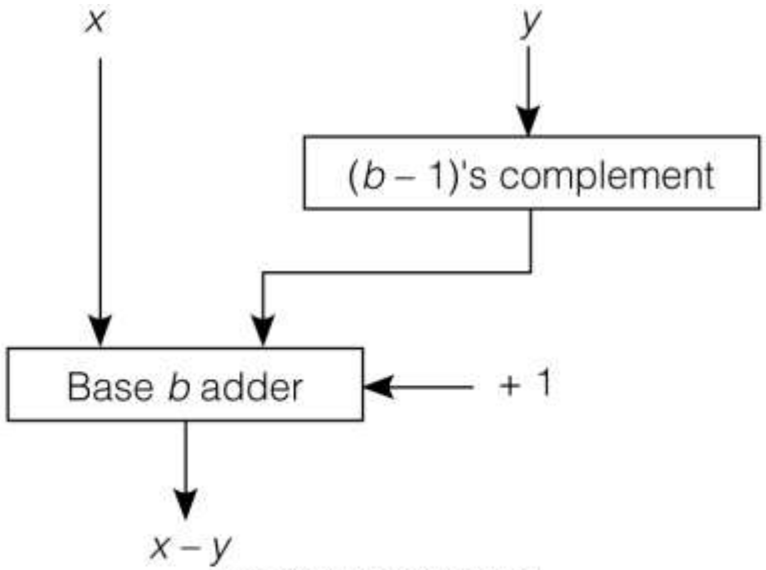
- For base $b=2^k$, each digit is equivalent to k bits
- Adder can be viewed as logic circuit with 2k+1 inputs and k+1 outputs



- Ripple carry adder
  - Choice of k affects computation delay
  - When 2 level logic is used what is max gate delay for m digit addition?
    - 2m

Complement Subtracter

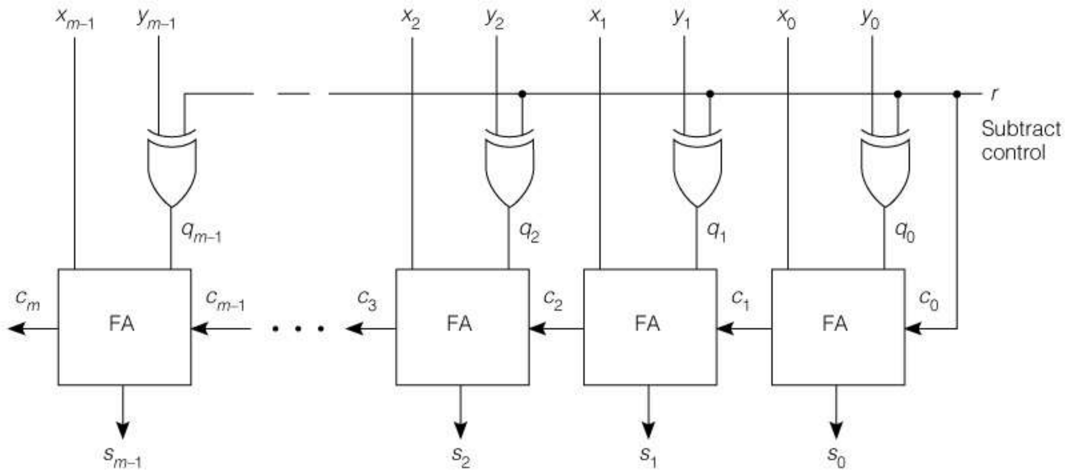
- Subtraction in radix complement is addition with negated (complemented) second input
  - Must supply overflow detection
- Radix complement is addition of 1 to diminished radix complement $(x^c = (\hat{x}^c + 1) \bmod b^m)$
  - Easy to take diminished radix complement and use carry in of adder to supply +1 for radix complement



Overflow Detection

- Occurs when adding number of like sign and the result seems to have opposite sign
- For even b: sign determined by the leftmost digit
  - Overflow detector only requires
    - x_{m-1}, y_{m-1}, s_{m-1}

Ripple-carry adder/subtracter



XOR gates select y for addition or complement of y for subtraction in base 2

Carry Lookahead

- Speed of addition depends on carries
  - Carries need to propagate from lsb to msb
- Two level logic for base b digit becomes complex quickly for increasing k ($b=2^k$)
  - Length of carry chain divided by k
- Need to compute carries quickly
  1. Determine if addition in position j generates a carry
  2. Determine if carry is propagated from input to output of digit j

Binary Generate and Propagate Signals

- Generate: digit at position j will have a carry
  - $G_j = x_j y_j$
- Propagate: carry in passes through to carry out
  - $P_j = x_j + y_j$
- Carry is defined as 1 if the sum generates a carry or if a carry is propagated
  - $c_{j+1} = G_j + P_j c_j$

Carry Lookahead Speed

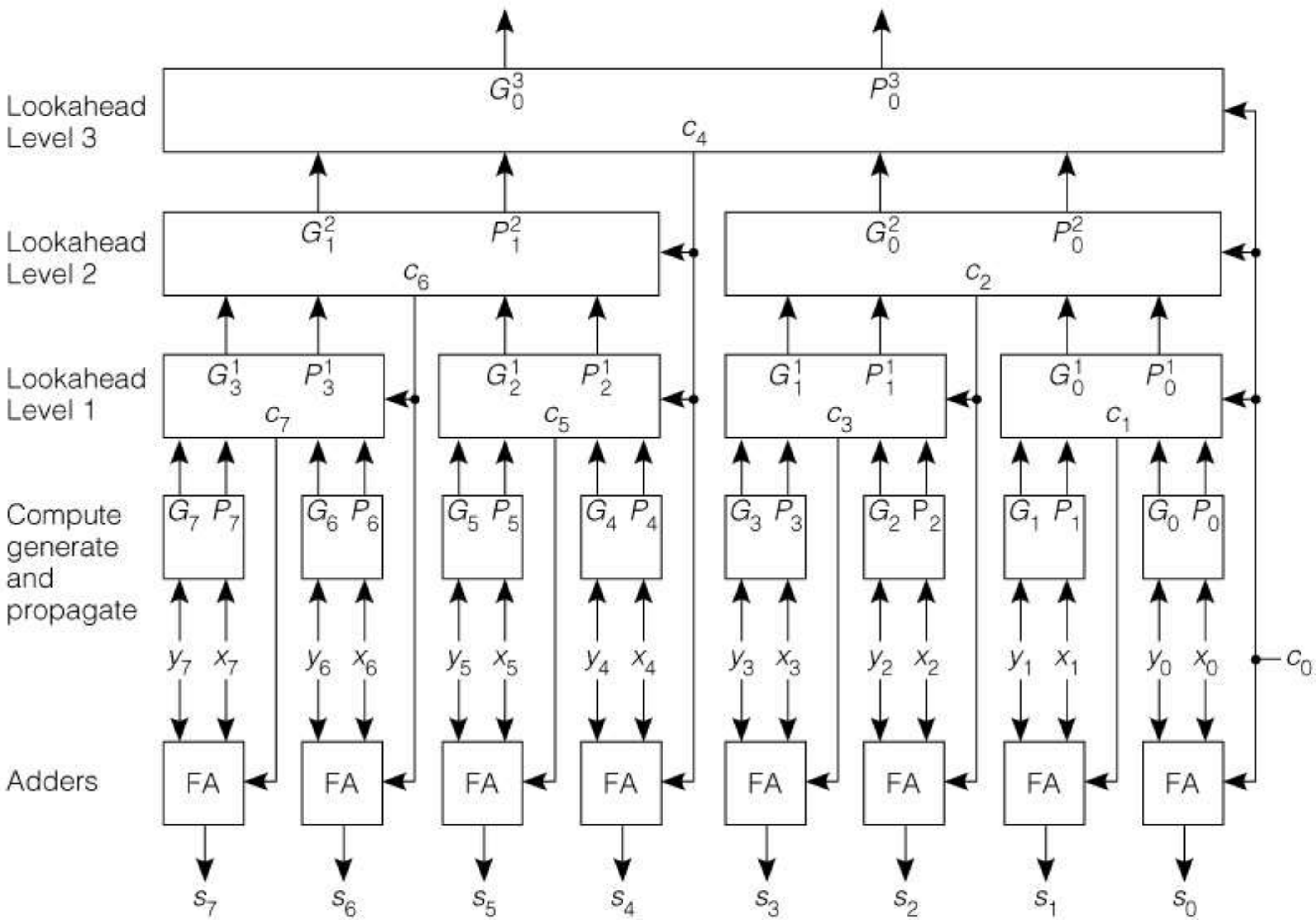
- 4 bit carry equations
  - $c_1 = G_0 + P_0c_0$
  - $c_2 = G_1 + P_1G_0 + P_1P_0c_0$
  - $c_3 = G_2 + P_2G_1 + P_2P_1G_0 + P_2P_1P_0c_0$
  - $c_4 = G_3 + P_3G_2 + P_3P_2G_1 + P_3P_2P_1G_0 + P_3P_2P_1P_0c_0$
- Carry lookahead delay
  - One gate delay for to calculate G or P
  - 2 levels of gates for a carry
  - 2 gate delays for full adder (s_j)
- The number of OR gate inputs (terms) and AND gate inputs (literals in a term) grows as the number of carries generated by lookahead

Recursive Carry Lookahead

- Apply lookahead logic to groups of digits
- Group of 4 digits (level 1)
  - Group generate:
    - $G^1{}_0 = G_3 + P_3G_2 + P_3P_2G_1 + P_3P_2P_1G_0$
  - Group propagate:
    - $P^1{}_0 = P_3P_2P_1P_0$
  - Can further define level 2 signals which are groups of level 1 groups
- Group k terms at each level → $\log_k m$ levels for m bit addition
  - Each level introduces 2 more gate delays
  - k chosen to trade-off reduced delay and complexity of G and P logic
    - Typically $k \geq 4$ however structure easier to see for k=2

Carry Lookahead Adder Diagram

- Group size k=2



Digital Multiplication

- Based on digital addition
  - Generate partial products (from each digit) and sum for the complete product
  - “Pencil and paper addition”

				x_3	x_2	x_1	x_0	Multiplicand
				y_3	y_2	y_1	y_0	Multiplier
			$(xy_0)_4$	$(xy_0)_3$	$(xy_0)_2$	$(xy_0)_1$	$(xy_0)_0$	pp_0
		$(xy_1)_4$	$(xy_1)_3$	$(xy_1)_2$	$(xy_1)_1$	$(xy_1)_0$		pp_1
	$(xy_2)_4$	$(xy_2)_3$	$(xy_2)_2$	$(xy_2)_1$	$(xy_2)_0$			pp_2
$(xy_3)_4$	$(xy_3)_3$	$(xy_3)_2$	$(xy_3)_1$	$(xy_3)_0$				pp_3
p_7	p_6	p_5	p_4	p_3	p_2	p_1	p_0	

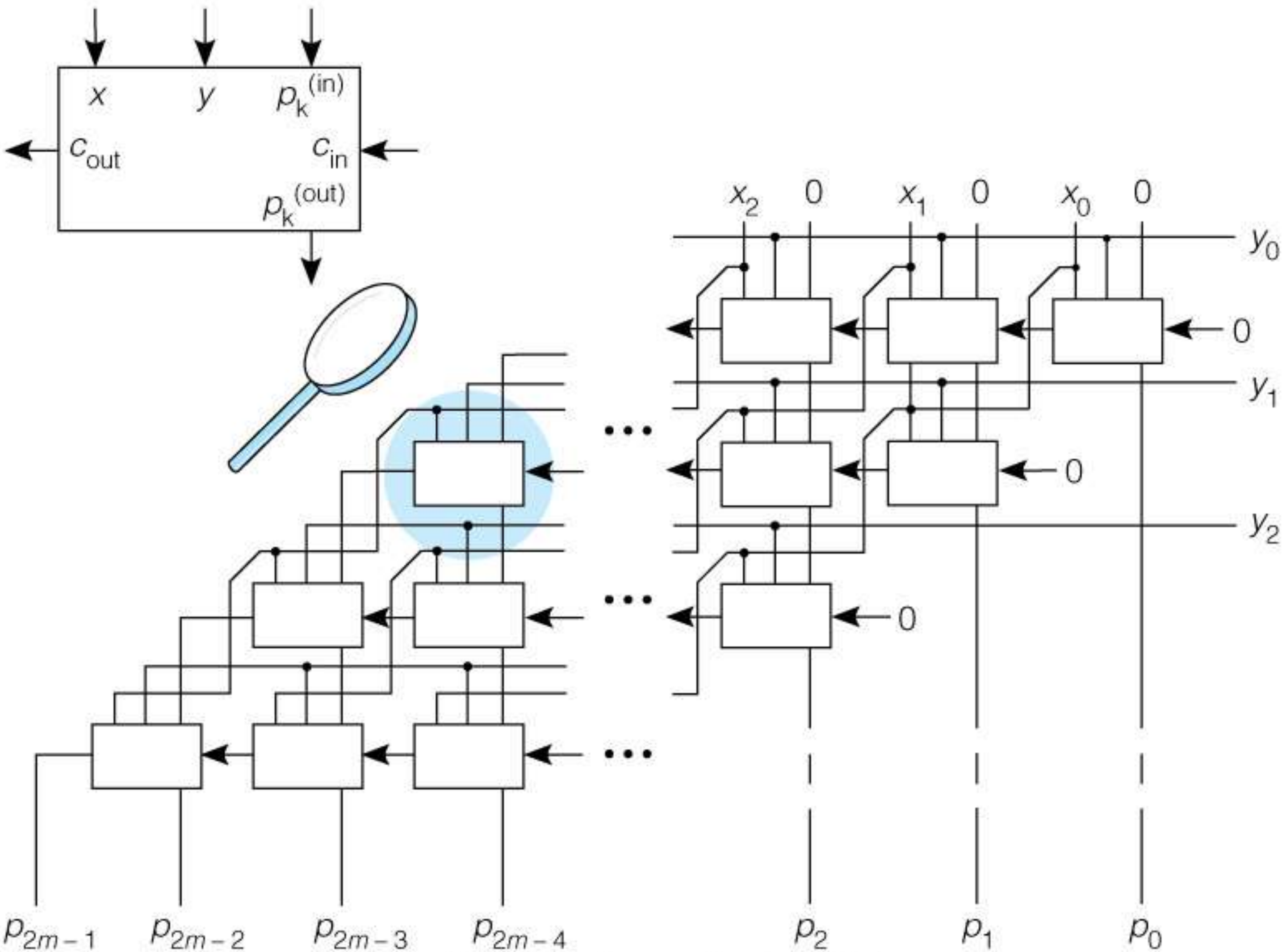
Accumulated Partial Product

- Partial products accumulated rather than collected and added in the end

for i := 0 step 1 until 2m-1
p_i := 0;
for j := 0 step 1 until m-1
begin
c := 0;
for i := 0 step 1 until m-1
begin
$p_{j+i} := (p_{j+i} + x_i\, y_j + c) \bmod b;$
$c := \lfloor (p_{j+i} + x_i\, y_j + c)/b \rfloor;$
end;
$p_{j+m} := c;$
end;

c is a single base b digit
(no longer 0, 1 as in addition)

Parallel Array Multiplier



Parallel Array Multiplier Operation

- Each box in array does the base b digit calculations
  - $p_k(\text{out}) := (p_k(\text{in}) + xy + c(\text{in})) \bmod b$
  - $c\ (\text{out}) := \lfloor (p_k(\text{in}) + xy + c)/b \rfloor$
- Inputs and outputs of boxes are single base b digits (including carries)
- Worst case path from input to output is about 6m gates if each box is a 2 level circuit
  - In binary, each box is a full adder with an extra AND gate to compute xy