

Chapter 2

Professor Brendan Morris, SEB 3216, brendan.morris@unlv.edu
<http://www.ee.unlv.edu/~b1morris/cpe100/>

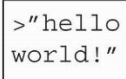
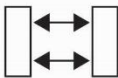
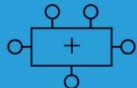
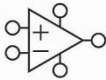
CPE100: Digital Logic Design I

Combinational Logic Design



Chapter 2 :: Topics

- Introduction
- Boolean Equations
- Boolean Algebra
- From Logic to Gates
- Multilevel Combinational Logic
- X's and Z's, Oh My
- Karnaugh Maps
- Combinational Building Blocks
- Timing

Application Software	
Operating Systems	
Architecture	
Micro-architecture	
Logic	
Digital Circuits	
Analog Circuits	
Devices	
Physics	

Chapter 2.1

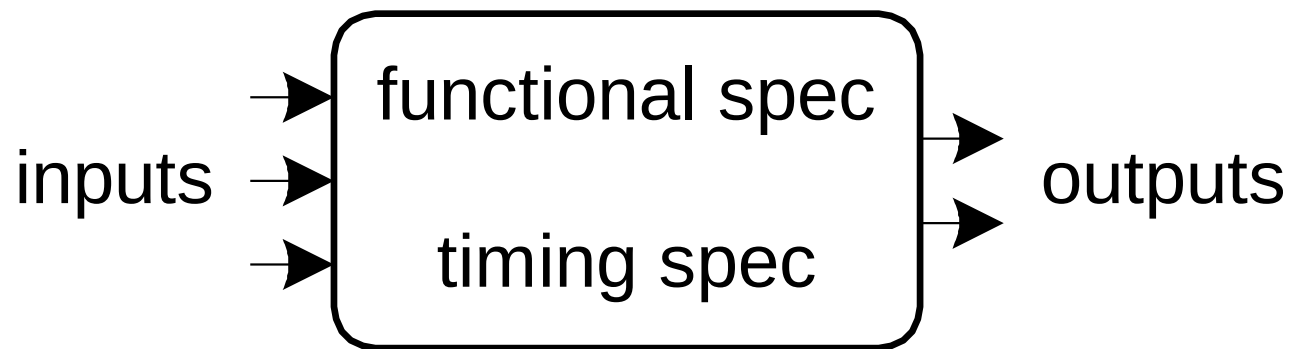
Introduction



Introduction

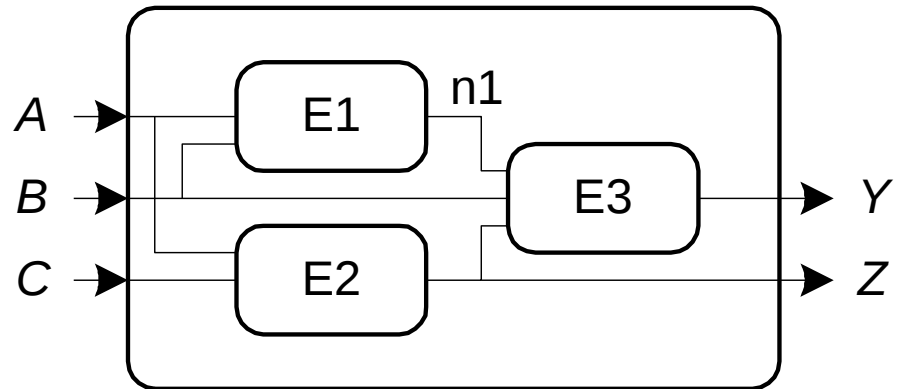
A logic circuit is composed of:

- Inputs
- Outputs
- Functional specification
- Timing specification



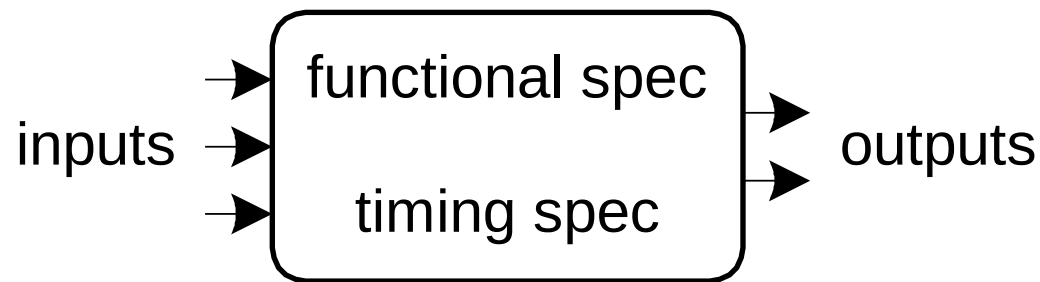
Circuits

- Nodes
 - Inputs: A, B, C
 - Outputs: Y, Z
 - Internal: $n1$
- Circuit elements
 - $E1, E2, E3$
 - Each a circuit



Types of Logic Circuits

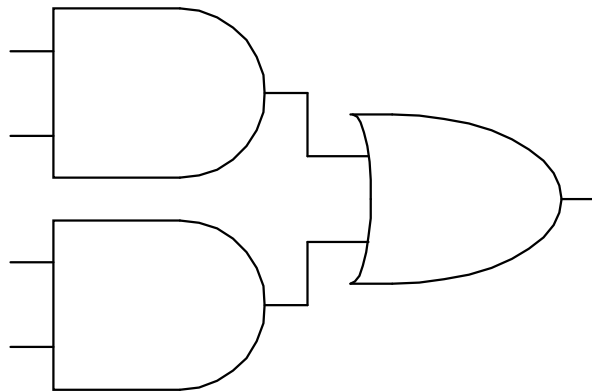
- Combinational Logic (Ch 2)
 - Memoryless
 - Outputs determined by current values of inputs
- Sequential Logic (Ch 3)
 - Has memory
 - Outputs determined by previous and current values of inputs



Rules of Combinational Composition

- Every element is combinational
- Every node is either an input or connects to *exactly one* output
- The circuit contains no cyclic paths
 - E.g. no connection from output to internal node

- Example:



Chapter 2.2

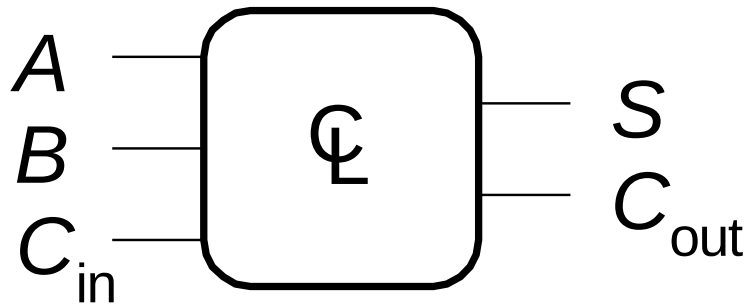
Boolean Equations



Boolean Equations

- Functional specification of outputs in terms of inputs

- Example:** $S = F(A, B, C_{in})$
 $C_{out} = F(A, B, C_{in})$



$$S = A \oplus B \oplus C_{in}$$

$$C_{out} = AB + AC_{in} + BC_{in}$$

A	B	C _{in}	S	C _{out}
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

Functional specification

Goals:

- Systematically express logical functions using Boolean equations
- To simplify Boolean equations

Some Definitions

- Complement: variable with a bar over it
 $\bar{A}, \bar{B}, \bar{C}$
- Literal: variable or its complement
 $A, \bar{A}, B, \bar{B}, C, \bar{C}$
- Implicant: product of literals
 $ABC, \bar{A}C, BC$
- Minterm: product that includes all input variables
 $ABC, \bar{A}\bar{B}\bar{C}, ABC$
- Maxterm: sum that includes all input variables
 $(A+\bar{B}+C), (\bar{A}+B+\bar{C}), (\bar{A}+\bar{B}+C)$



Canonical Sum-of-Products (SOP) Form

- All equations can be written in SOP form
- Each row has a **minterm**
- A minterm is a product (AND) of literals
- Each minterm is TRUE for that row (and only that row)

<i>A</i>	<i>B</i>	<i>Y</i>	minterm	minterm name
0	0	0	$\overline{A} \overline{B}$	m_0
0	1	1	$\overline{A} B$	m_1
1	0	0	$A \overline{B}$	m_2
1	1	1	$A B$	m_3

Canonical Sum-of-Products (SOP) Form

- All equations can be written in SOP form
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- Form function by ORing minterms where the output is TRUE

<i>A</i>	<i>B</i>	<i>Y</i>	minterm	minterm name
0	0	0	$\overline{A} \overline{B}$	m_0
0	1	1	$\overline{A} B$	m_1
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1	1	1	$A B$	m_3

$$Y = F(A, B) =$$

Canonical Sum-of-Products (SOP) Form

- All equations can be written in SOP form
- Each row has a **minterm**
- A minterm is a product (AND) of literals
- Each minterm is TRUE for that row (and only that row)
- Form function by ORing minterms where the output is TRUE
- Thus, a sum (OR) of products (AND terms)

A	B	Y	minterm	minterm name
0	0	0	$\overline{A} \overline{B}$	m_0
0	1	1	$\overline{A} B$	m_1
1	0	0	$A \overline{B}$	m_2
1	1	1	$A B$	m_3

$$Y = F(A, B) = \overline{A}B + AB = \Sigma(m_1, m_3)$$

SOP Example

- Steps:
- Find minterms that result in $Y=1$
- Sum “TRUE” minterms

A	B	Y
0	0	1
0	1	1
1	0	0
1	1	0

$$Y = F(A, B) =$$

SOP Example

- Steps:
- Find minterms that result in $Y=1$
- Sum “TRUE” minterms

A	B	Y	minterm name	minterm
0	0	1	m_0	$\bar{A}\bar{B}$
0	1	1	m_1	$\bar{A}B$
1	0	0	m_2	
1	1	0	m_3	

$$Y = F(A, B) = m_0 + m_1 = \bar{A}\bar{B} + \bar{A}B$$

Aside: Precedence

- AND has precedence over OR
- In other words:
 - AND is performed **before** OR
- Example:
 - $Y = \bar{A} \cdot B + A \cdot B$
 - Equivalent to:
 - $Y = (\bar{A}B) + (AB)$

Canonical Product-of-Sums (POS) Form

- All Boolean equations can be written in POS form
- Each row has a **maxterm**
- A maxterm is a sum (OR) of literals
- Each maxterm is FALSE for that row (and only that row)

<i>A</i>	<i>B</i>	<i>Y</i>	maxterm	maxterm name
0	0	0	$A + B$	M_0
0	1	1	$A + \overline{B}$	M_1
1	0	0	$\overline{A} + B$	M_2
1	1	1	$\overline{A} + \overline{B}$	M_3

Canonical Product-of-Sums (POS) Form

- All Boolean equations can be written in POS form
- Each row has a **maxterm**
- A maxterm is a sum (OR) of literals
- Each maxterm is FALSE for that row (and only that row)
- Form function by ANDing the maxterms for which the output is FALSE
- Thus, a product (AND) of sums (OR terms)

A	B	Y	maxterm	maxterm name
0	0	0	$A + B$	M_0
0	1	1	$A + \overline{B}$	M_1
1	0	0	$\overline{A} + B$	M_2
1	1	1	$\overline{A} + \overline{B}$	M_3

$$Y = M_0 \cdot M_2 = (A + B) \cdot (\overline{A} + B)$$

SOP and POS Comparison

- Sum of Products (SOP)
 - Implement the “ones” of the output
 - Sum all “one” terms $\rightarrow$ OR results in “one”
- Product of Sums (POS)
 - Implement the “zeros” of the output
 - Multiply “zero” terms $\rightarrow$ AND results in “zero”



Boolean Equations Example

- You are going to the cafeteria for lunch
 - You will eat lunch ($E=1$)
 - If it's open ($O=1$) **and**
 - If they're not serving corndogs ($C=0$)
- Write a truth table for determining if you will eat lunch (E).

O	C	E
0	0	
0	1	
1	0	
1	1	

Boolean Equations Example

- You are going to the cafeteria for lunch
 - You will eat lunch ($E=1$)
 - If it's open ($O=1$) **and**
 - If they're not serving corndogs ($C=0$)
- Write a truth table for determining if you will eat lunch (E).

O	C	E
0	0	0
0	1	0
1	0	1
1	1	0

SOP & POS Form

- SOP – sum-of-products

O	C	E	minterm
0	0		$\overline{O} \overline{C}$
0	1		$\overline{O} C$
1	0		$O \overline{C}$
1	1		$O C$

- POS – product-of-sums

O	C	E	maxterm
0	0		$O + C$
0	1		$O + \overline{C}$
1	0		$\overline{O} + C$
1	1		$\overline{O} + \overline{C}$

SOP & POS Form

- SOP – sum-of-products

O	C	E	minterm
0	0	0	$\overline{O} \overline{C}$
0	1	0	$\overline{O} C$
1	0	1	$O \overline{C}$
1	1	0	$O C$

- POS – product-of-sums

O	C	E	maxterm
0	0	0	$O + C$
0	1	0	$O + \overline{C}$
1	0	1	$\overline{O} + C$
1	1	0	$\overline{O} + \overline{C}$

SOP & POS Form

- SOP – sum-of-products

O	C	E	minterm
0	0	0	$\overline{O} \overline{C}$
0	1	0	$\overline{O} C$
1	0	1	$O \overline{C}$
1	1	0	$O C$

$$E = O\overline{C}$$

$$= \Sigma(m_2)$$

- POS – product-of-sums

O	C	E	maxterm
0	0	0	$O + C$
0	1	0	$O + \overline{C}$
1	0	1	$\overline{O} + C$
1	1	0	$\overline{O} + \overline{C}$

SOP & POS Form

- SOP – sum-of-products

O	C	E	minterm
0	0	0	$\overline{O} \overline{C}$
0	1	0	$\overline{O} C$
1	0	1	$O \overline{C}$
1	1	0	$O C$

$$E = O\overline{C}$$

$$= \Sigma(m_2)$$

- POS – product-of-sums

O	C	E	maxterm
0	0	0	$O + C$
0	1	0	$O + \overline{C}$
1	0	1	$\overline{O} + C$
1	1	0	$\overline{O} + \overline{C}$

$$E = (O + C)(O + \overline{C})(\overline{O} + \overline{C})$$

$$= \Pi(M_0, M_1, M_3)$$

Chapter 2.3

Boolean Algebra



Boolean Algebra

- Axioms and theorems to **simplify** Boolean equations
- Like regular algebra, but simpler: variables have only two values (1 or 0)
- **Duality** in axioms and theorems:
 - ANDs and ORs, 0's and 1's interchanged

Boolean Axioms

Axiom

$$A1 \quad B = 0 \text{ if } B \neq 1$$

$$A2 \quad \overline{0} = 1$$

$$A3 \quad 0 \bullet 0 = 0$$

$$A4 \quad 1 \bullet 1 = 1$$

$$A5 \quad 0 \bullet 1 = 1 \bullet 0 = 0$$

Duality

Duality in Boolean axioms and theorems:

- ANDs and ORs, 0's and 1's interchanged

Boolean Axioms

Axiom	
A1	$B = 0 \text{ if } B \neq 1$
A2	$\overline{0} = 1$
A3	$0 \bullet 0 = 0$
A4	$1 \bullet 1 = 1$
A5	$0 \bullet 1 = 1 \bullet 0 = 0$

Boolean Axioms

Axiom		Dual	
A1	$B = 0 \text{ if } B \neq 1$	A1'	$B = 1 \text{ if } B \neq 0$
A2	$\overline{0} = 1$	A2'	$\overline{1} = 0$
A3	$0 \bullet 0 = 0$	A3'	$1 + 1 = 1$
A4	$1 \bullet 1 = 1$	A4'	$0 + 0 = 0$
A5	$0 \bullet 1 = 1 \bullet 0 = 0$	A5'	$1 + 0 = 0 + 1 = 1$

Dual: Exchange: $\bullet$ and $+$
 0 and 1

Boolean Axioms

	Axiom		Dual	Name
A1	$B = 0 \text{ if } B \neq 1$	A1'	$B = 1 \text{ if } B \neq 0$	Binary field
A2	$\overline{0} = 1$	A2'	$\overline{1} = 0$	NOT
A3	$0 \bullet 0 = 0$	A3'	$1 + 1 = 1$	AND/OR
A4	$1 \bullet 1 = 1$	A4'	$0 + 0 = 0$	AND/OR
A5	$0 \bullet 1 = 1 \bullet 0 = 0$	A5'	$1 + 0 = 0 + 1 = 1$	AND/OR

Dual: Exchange: $\bullet$ and $+$
0 and 1

Basic Boolean Theorems

Theorem	
T1	$B \bullet 1 = B$
T2	$B \bullet 0 = 0$
T3	$B \bullet B = B$
T4	$\overline{\overline{B}} = B$
T5	$B \bullet \overline{B} = 0$

Basic Boolean Theorems: Duals

	Theorem		Dual	Name
T1	$B \bullet 1 = B$	T1'	$B + 0 = B$	Identity
T2	$B \bullet 0 = 0$	T2'	$B + 1 = 1$	Null Element
T3	$B \bullet B = B$	T3'	$B + B = B$	Idempotency
T4		$\overline{\overline{B}} = B$		Involution
T5	$B \bullet \overline{B} = 0$	T5'	$B + \overline{B} = 1$	Complements

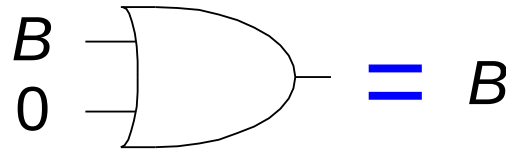
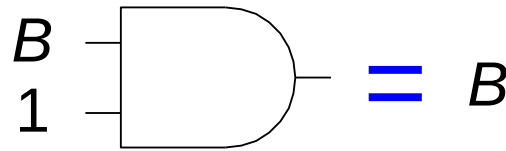
Dual: Exchange: $\bullet$ and $+$
 0 and 1

T1: Identity Theorem

- $B \cdot 1 = B$
- $B + 0 = B$

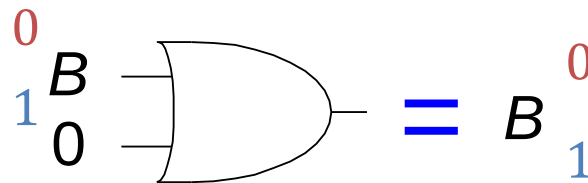
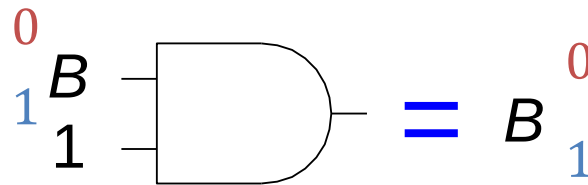
T1: Identity Theorem

- $B \cdot 1 = B$
- $B + 0 = B$



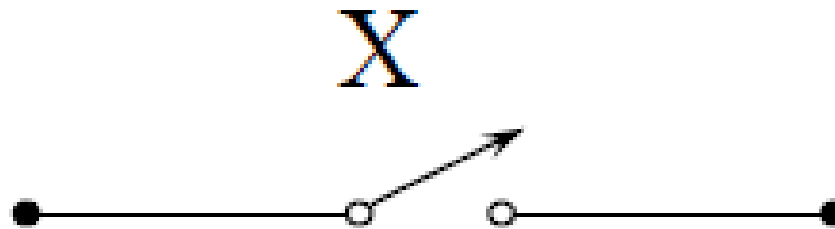
T1: Identity Theorem

- $B \cdot 1 = B$
- $B + 0 = B$



Switching Algebra

- Simplification of digital logic → connecting wires with a on/off switch
- $X = 0$ (switch open)
- $X = 1$ (switch closed)



Series Switching Network: AND

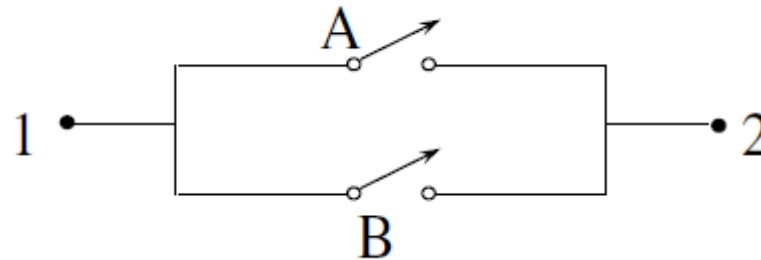
- Switching circuit in series performs AND



- 1 is connected to 2 iff A **AND** B are 1

Parallel Switching Network: OR

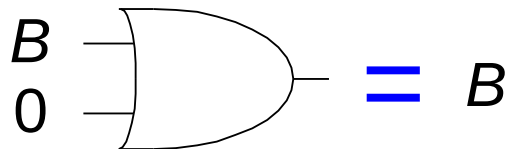
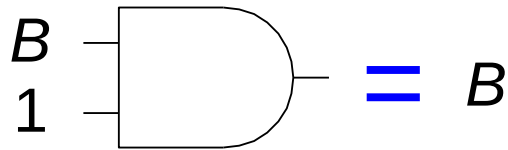
- Switching circuit in parallel performs OR



- 1 is connected to 2 if A **OR** B is 1

T1: Identity Theorem

- $B \cdot 1 = B$
- $B + 0 = B$

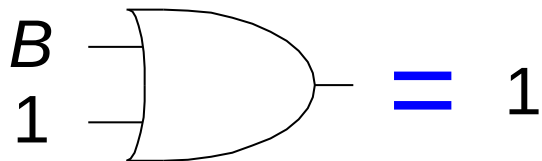
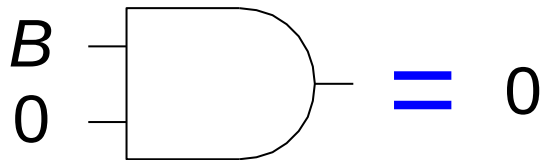


T2: Null Element Theorem

- $B \cdot 0 = 0$
- $B + 1 = 1$

T2: Null Element Theorem

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- $B + 1 = 1$

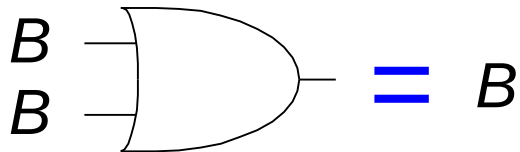
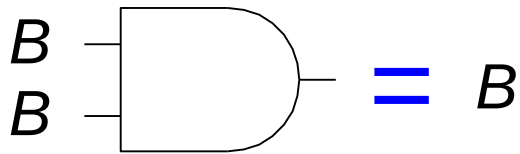


T3: Idempotency Theorem

- $B \cdot B = B$
- $B + B = B$

T3: Idempotency Theorem

- $B \cdot B = B$
- $B + B = B$

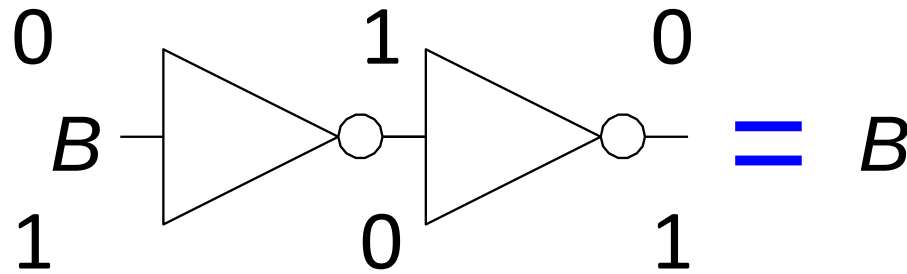


T4: Involution Theorem

- $\overline{\overline{B}} = B$

T4: Involution Theorem

- $\overline{\overline{B}} = B$

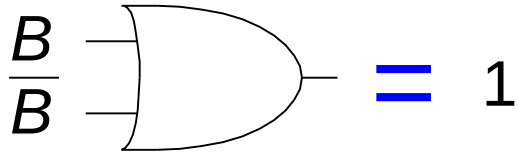
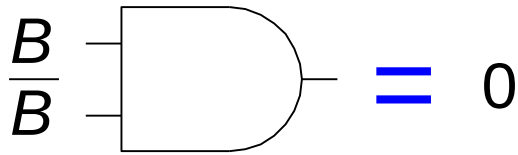


T5: Complements Theorem

- $B \cdot \overline{B} = 0$
- $B + \overline{B} = 1$

T5: Complements Theorem

- $B \cdot \bar{B} = 0$
- $B + \bar{B} = 1$



Recap: Basic Boolean Theorems

	Theorem		Dual	Name
T1	$B \bullet 1 = B$	T1'	$B + 0 = B$	Identity
T2	$B \bullet 0 = 0$	T2'	$B + 1 = 1$	Null Element
T3	$B \bullet B = B$	T3'	$B + B = B$	Idempotency
T4		$\overline{\overline{B}} = B$		Involution
T5	$B \bullet \overline{B} = 0$	T5'	$B + \overline{B} = 1$	Complements

Chapter 2.3.3

Theorems of Several Variables



Boolean Theorems of Several Vars

Number	Theorem	Name
T6	$B \bullet C = C \bullet B$	Commutativity
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	Associativity
T8	$B \bullet (C + D) = (B \bullet C) + (B \bullet D)$	Distributivity
T9	$B \bullet (B + C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \bar{C}) = B$	Combining
T11	$B \bullet C + (\bar{B} \bullet D) + (C \bullet D) = B \bullet C + \bar{B} \bullet D$	Consensus

Boolean Theorems of Several Vars

Number	Theorem	Name
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T8	$B \bullet (C + D) = (B \bullet C) + (B \bullet D)$	Distributivity
T9	$B \bullet (B + C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \bar{C}) = B$	Combining
T11	$B \bullet C + (\bar{B} \bullet D) + (C \bullet D) = B \bullet C + \bar{B} \bullet D$	Consensus

How do we prove these are true?

How to Prove Boolean Relation

- **Method 1:** Perfect induction
- **Method 2:** Use other theorems and axioms to simplify the equation
 - Make one side of the equation look like the other

Proof by Perfect Induction

- Also called: proof by exhaustion
- Check every possible input value
- If two expressions produce the same value for every possible input combination, the expressions are equal

Example: Proof by Perfect Induction

Number	Theorem	Name
T6	$B \bullet C = C \bullet B$	Commutativity

B	C	BC	CB
0	0		
0	1		
1	0		
1	1		

Example: Proof by Perfect Induction

Number	Theorem	Name
T6	$B \bullet C = C \bullet B$	Commutativity

B	C	BC	CB
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

Boolean Theorems of Several Vars

Number	Theorem	Name
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T11	$B \bullet C + (\bar{B} \bullet D) + (C \bullet D) = B \bullet C + \bar{B} \bullet D$	Consensus

T7: Associativity

Number	Theorem	Name
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	Associativity

T8: Distributivity

Number	Theorem	Name
T8	$B \bullet (C + D) = (B \bullet C) + (B \bullet D)$	Distributivity

T9: Covering

Number	Theorem	Name
T9	$B \bullet (B+C) = B$	Covering

T9: Covering

Number	Theorem	Name
T9	$B \bullet (B+C) = B$	Covering

Prove true by:

- **Method 1:** Perfect induction
- **Method 2:** Using other theorems and axioms

T9: Covering

Number	Theorem	Name
T9	$B \bullet (B+C) = B$	Covering

Method 1: Perfect Induction

B	C	$(B+C)$	$B(B+C)$
0	0		
0	1		
1	0		
1	1		

T9: Covering

Number	Theorem	Name
T9	$B \bullet (B+C) = B$	Covering

Method 1: Perfect Induction

B	C	$(B+C)$	$B(B+C)$
0	0	0	0
0	1	1	0
1	0	1	1
1	1	1	1

T9: Covering

Number	Theorem	Name
T9	$B \bullet (B+C) = B$	Covering

Method 2: Prove true using other axioms and theorems.

T9: Covering

Number	Theorem	Name
T9	$B \bullet (B+C) = B$	Covering

Method 2: Prove true using other axioms and theorems.

$$\begin{aligned} B \bullet (B+C) &= B \bullet B + B \bullet C \\ &= \mathbf{B} + B \bullet C \\ &= B \bullet (1 + C) \\ &= B \bullet (\mathbf{1}) \\ &= B \end{aligned}$$

T8: Distributivity

T3: Idempotency

T8: Distributivity

T2: Null element

T1: Identity

T10: Combining

Number	Theorem	Name
T10	$(B \bullet C) + (B \bullet \bar{C}) = B$	Combining

Prove true using other axioms and theorems:

T10: Combining

Number	Theorem	Name
T10	$(B \bullet C) + (B \bullet \bar{C}) = B$	Combining

Prove true using other axioms and theorems:

$$\begin{aligned} B \bullet C + B \bullet \bar{C} &= B \bullet (C + \bar{C}) && \text{T8: Distributivity} \\ &= B \bullet (1) && \text{T5': Complements} \\ &= B && \text{T1: Identity} \end{aligned}$$

T11: Consensus

Number	Theorem	Name
T11	$(B \bullet C) + (\bar{B} \bullet D) + (C \bullet D) = (B \bullet C) + \bar{B} \bullet D$	Consensus

Prove true using (1) perfect induction or (2) other axioms and theorems.



Recap: Boolean Thms of Several Vars

Number	Theorem	Name
T6	$B \bullet C = C \bullet B$	Commutativity
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	Associativity
T8	$B \bullet (C + D) = (B \bullet C) + (B \bullet D)$	Distributivity
T9	$B \bullet (B + C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \bar{C}) = B$	Combining
T11	$B \bullet C + (\bar{B} \bullet D) + (C \bullet D) = B \bullet C + \bar{B} \bullet D$	Consensus

Boolean Thms of Several Vars: Duals

#	Theorem	Dual	Name
T6	$B \bullet C = C \bullet B$	$B + C = C + B$	Commutativity
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	$(B + C) + D = B + (C + D)$	Associativity
T8	$B \bullet (C + D) = (B \bullet C) + (B \bullet D)$	$B + (C \bullet D) = (B + C) (B + D)$	Distributivity
T9	$B \bullet (B + C) = B$	$B + (B \bullet C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \bar{C}) = B$	$(B + C) \bullet (B + \bar{C}) = B$	Combining
T11	$(B \bullet C) + (B \bullet \bar{D}) + (C \bullet D) = (B \bullet C) + (B \bullet \bar{D})$	$(B + C) \bullet (B + \bar{D}) \bullet (C + D) = (B + C) \bullet (B + \bar{D})$	Consensus

Dual: Replace: $\bullet$ with $+$
 0 with 1

Boolean Thms of Several Vars: Duals

#	Theorem	Dual	Name
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T8	$B \bullet (C + D) = (B \bullet C) + (B \bullet D)$	$B + (C \bullet D) = (B + C) (B + D)$	Distributivity
T9	$B \bullet (B + C) = B$	$B + (B \bullet C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \bar{C}) = B$	$(B + C) \bullet (B + \bar{C}) = B$	Combining
T11	$(B \bullet C) + (B \bullet \bar{D}) + (C \bullet D) = (B \bullet C) + (B \bullet \bar{D})$	$(B + C) \bullet (B + \bar{D}) \bullet (C + D) = (B + C) \bullet (B + \bar{D})$	Consensus

Dual: Replace: $\bullet$ with $+$
 0 with 1

Warning: T8' differs from traditional algebra: OR ($+$) distributes over AND ($\bullet$)



Boolean Thms of Several Vars: Duals

#	Theorem	Dual	Name
T6	$B \bullet C = C \bullet B$	$B + C = C + B$	Commutativity
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T9	$B \bullet (B + C) = B$	$B + (B \bullet C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	$(B + C) \bullet (B + \overline{C}) = B$	Combining
T11	$(B \bullet C) + (B \bullet \overline{D}) + (C \bullet D) = (B \bullet C) + (B \bullet \overline{D})$	$(B + C) \bullet (B + \overline{D}) \bullet (C + D) = (B + C) \bullet (B + \overline{D})$	Consensus

Axioms and theorems are useful for *simplifying* equations.

Chapter 2.3.5

Simplifying Equations



Simplifying an Equation

- Reducing an equation to the fewest number of implicants, where each implicant has the fewest literals

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Recall:

- **Implicant:** product of literals
 $ABC, AC, \bar{B}C$
- **Literal:** variable or its complement
 $A, \bar{A}, B, \bar{B}, C, \bar{C}$

Simplifying an Equation

- Reducing an equation to the fewest number of implicants, where each implicant has the fewest literals

Recall:

- **Implicant:** product of literals
 $ABC, AC, \bar{B}C$
- **Literal:** variable or its complement
 $A, \bar{A}, B, \bar{B}, C, \bar{C}$

- Also called: **minimizing** the equation

Simplification methods

- **Distributivity (T8, T8')**

$$B(C+D) = BC + BD$$

$$B + CD = (B + C)(B + D)$$

- **Covering (T9')**

$$A + AP = A$$

- **Combining (T10)**

$$PA + \overline{PA} = P$$

P	A	Minterm
0	0	$\overline{P}\overline{A}$
0	1	$\overline{P}A$
1	0	$P\overline{A}$
1	1	PA

Simplification methods

- Distributivity (T8, T8')**

$$B(C+D) = BC + BD$$

$$B + CD = (B + C)(B + D)$$

- Covering (T9')**

$$A + AP = A$$

- Combining (T10)**

$$PA + \overline{PA} = P$$

- Expansion**

$$P = PA + \overline{PA}$$

$$A = A + AP$$

- Duplication**

$$A = A + A$$

P	A	Minterm
0	0	$\overline{P}\overline{A}$
0	1	$\overline{P}A$
1	0	$P\overline{A}$
1	1	PA

Simplification methods

- **Distributivity (T8, T8')**

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- **Expansion**

$$P = PA + \overline{PA}$$

$$A = A + AP$$

- **Duplication**

$$A = A + A$$

- **A combination of Combining/Covering**

$$PA + \overline{A} = P + \overline{A}$$

Simplification methods

- **A combination of Combining/Covering**

$$PA + \bar{A} = P + \bar{A}$$

Proof:

$$\begin{aligned} PA + \bar{A} &= PA + (\bar{A} + \bar{A}P) \\ &= PA + P\bar{A} + \bar{A} \\ &= P(A + \bar{A}) + \bar{A} \\ &= P(1) + \bar{A} \\ &= P + \bar{A} \end{aligned}$$

T9' Covering

T6 Commutativity

T8 Distributivity

T5' Complements

T1 Identity

Recap: Boolean Thms of Several Vars

#	Theorem	Dual	Name
T6	$B \bullet C = C \bullet B$	$B + C = C + B$	Commutativity
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	$(B + C) + D = B + (C + D)$	Associativity
T8	$B \bullet (C + D) = (B \bullet C) + (B \bullet D)$	$B + (C \bullet D) = (B + C) (B + D)$	Distributivity
T9	$B \bullet (B + C) = B$	$B + (B \bullet C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	$(B + C) \bullet (B + \overline{C}) = B$	Combining
T11	$(B \bullet C) + (B \bullet \overline{D}) + (C \bullet D) = (B \bullet C) + (B \bullet \overline{D})$	$(B + C) \bullet (B + \overline{D}) \bullet (C + D) = (B + C) \bullet (B + \overline{D})$	Consensus

T11: Consensus

Number	Theorem	Name
T11	$(B \bullet C) + (\bar{B} \bullet D) + (C \bullet D) = (B \bullet C) + \bar{B} \bullet D$	Consensus

Prove using other theorems and axioms:

T11: Consensus

Number	Theorem	Name
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D) = (B \bullet C) + \overline{B} \bullet D$	Consensus

Prove using other theorems and axioms:

$$\begin{aligned} B \bullet C + \overline{B} \bullet D + C \bullet D \\ &= BC + \overline{BD} + (CDB + C\overline{D}\overline{B}) \\ &= BC + \overline{BD} + BCD + \overline{B}C\overline{D} \\ &= BC + BCD + \overline{BD} + \overline{B}C\overline{D} \\ &= (BC + BCD) + (\overline{BD} + \overline{B}C\overline{D}) \\ &= BC + \overline{BD} \end{aligned}$$

T10: Combining
T6: Commutativity
T6: Commutativity
T7: Associativity
T9': Covering

Simplification methods

- **Distributivity (T8, T8')**

$$B(C+D) = BC + BD$$

$$B + CD = (B + C)(B + D)$$

- **Covering (T9')**

$$A + AP = A$$

- **Combining (T10)**

$$PA + \overline{P}\overline{A} = P$$

- **Expansion**

$$P = PA + \overline{P}\overline{A}$$

$$A = A + AP$$

- **Duplication**

$$A = A + A$$

- **A combination of Combining/Covering**

$$PA + \overline{P}\overline{A} = P + \overline{A}$$

Simplifying Boolean Equations

Example 1:

$$Y = AB + A\overline{B}$$

Simplifying Boolean Equations

Example 1:

$$Y = AB + A\bar{B}$$

$$Y = A$$

T10: Combining

or

$$= A(B + \bar{B})$$

T8: Distributivity

$$= A(1)$$

T5': Complements

$$= A$$

T1: Identity



Simplification methods

- **Distributivity (T8, T8')**

$$B(C+D) = BC + BD$$

$$B + CD = (B + C)(B + D)$$

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- **A combination of Combining/Covering**

$$PA + \overline{P}\overline{A} = P + \overline{A}$$

Simplifying Boolean Equations

Example 2:

$$Y = A(AB + ABC)$$

Simplifying Boolean Equations

Example 2:

$$Y = A(AB + ABC)$$

$$= A(AB(1 + C))$$

$$= A(AB(1))$$

$$= A(AB)$$

$$= (AA)B$$

$$= AB$$

T8: Distributivity

T2': Null Element

T1: Identity

T7: Associativity

T3: Idempotency

Simplification methods

- **Distributivity (T8, T8')**

$$B(C+D) = BC + BD$$

$$B + CD = (B + C)(B + D)$$

- **Covering (T9')**

$$A + AP = A$$

- **Combining (T10)**

$$PA + \overline{PA} = P$$

- **Expansion**

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$$A = A + AP$$

- **Duplication**

$$A = A + A$$

- **A combination of Combining/Covering**

$$PA + \overline{A} = P + \overline{A}$$

Simplifying Boolean Equations

Example 3:

$$Y = A'BC + A'$$

Recall: $A' = \overline{A}$

Simplifying Boolean Equations

Example 3:

$$Y = A'BC + A'$$

$$= A'$$

or

$$= A'(BC + 1)$$

$$= A'(1)$$

$$= A'$$

$$\text{Recall: } A' = \overline{A}$$

$$\text{T9' Covering: } X + XY = X$$

T8: Distributivity

T2': Null Element

T1: Identity

Simplification methods

- **Distributivity (T8, T8')**

$$B(C+D) = BC + BD$$

$$B + CD = (B + C)(B + D)$$

- **Covering (T9')**

$$A + AP = A$$

- **Combining (T10)**

$$PA + \overline{PA} = P$$

- **Expansion**

$$P = PA + \overline{PA}$$

$$A = A + AP$$

- **Duplication**

$$A = A + A$$

- **A combination of Combining/Covering**

$$PA + \overline{A} = P + \overline{A}$$

Simplifying Boolean Equations

Example 4:

$$Y = AB'C + ABC + A'BC$$

Simplifying Boolean Equations

Example 4:

$$Y = AB'C + ABC + A'BC$$

$$= AB'C + \mathbf{ABC} + \mathbf{ABC} + A'BC \quad \text{T3': Idempotency}$$

$$= (AB'C + ABC) + (ABC + A'BC) \quad \text{T7': Associativity}$$

$$= AC + BC \quad \text{T10: Combining}$$

Simplification methods

- **Distributivity (T8, T8')**

$$B(C+D) = BC + BD$$

$$B + CD = (B + C)(B + D)$$

- **Covering (T9')**

$$A + AP = A$$

- **Combining (T10)**

$$PA + \overline{P}\overline{A} = P$$

- **Expansion**

$$P = PA + \overline{P}\overline{A}$$

$$A = A + AP$$

- **Duplication**

$$A = A + A$$

- **A combination of Combining/Covering**

$$PA + \overline{A} = P + \overline{A}$$

Simplifying Boolean Equations

Example 5:

$$Y = AB + BC + B'D' + AC'D'$$

Simplifying Boolean Equations

Example 5:

$$Y = AB + BC + B'D' + AC'D'$$

Method 1:

$$\begin{aligned} Y &= AB + BC + B'D' + (ABC'D' + AB'C'D') \\ &= (AB + ABC'D') + BC + (B'D' + AB'C'D') \\ &= AB + BC + B'D' \end{aligned}$$

T10: Combining
T6: Commutativity
T7: Associativity
T9: Covering

Simplifying Boolean Equations

Example 5:

$$Y = AB + BC + B'D' + AC'D'$$

Method 1:

$$\begin{aligned} Y &= AB + BC + B'D' + (ABC'D' + AB'C'D') \\ &= (AB + ABC'D') + BC + (B'D' + AB'C'D') \end{aligned}$$

$$= AB + BC + B'D'$$

Method 2:

$$\begin{aligned} Y &= \mathbf{AB} + BC + \mathbf{B'D'} + AC'D' + \mathbf{AD'} \\ &= AB + BC + B'D' + \mathbf{AD'} \\ &= AB + BC + B'D' \end{aligned}$$

T10: Combining

T6: Commutativity

T7: Associativity

T9: Covering

T11: Consensus

T9: Covering

T11: Consensus



Simplification methods

- **Distributivity (T8, T8')**

$$B(C+D) = BC + BD$$

$$B + CD = (B + C)(B + D)$$

- **Covering (T9')**

$$A + AP = A$$

- **Combining (T10)**

$$PA + \overline{P}\overline{A} = P$$

- **Expansion**

$$P = PA + \overline{P}\overline{A}$$

$$A = A + AP$$

- **Duplication**

$$A = A + A$$

- **A combination of Combining/Covering**

$$PA + \overline{P}\overline{A} = P + \overline{A}$$

Simplifying Boolean Equations

Example 6:

$$Y = (A + BC)(A + DE)$$

Apply T8' first when possible: $W + XZ = (W + X)(W + Z)$

Simplifying Boolean Equations

Example 6:

$$Y = (A + BC)(A + DE)$$

Apply T8' first when possible: $W + XZ = (W + X)(W + Z)$

Make: $X = BC$, $Z = DE$ and rewrite equation

$$Y = (A + X)(A + Z)$$

substitution ($X = BC$, $Z = DE$)

$$= A + XZ$$

T8': Distributivity

$$= A + BCDE$$

substitution

or

$$Y = AA + ADE + ABC + BCDE$$

T8: Distributivity

$$= A + ADE + ABC + BCDE$$

T3: Idempotency

$$= A + ADE + ABC + BCDE$$

$$= A + ABC + BCDE$$

T9': Covering

$$= A + BCDE$$

T9': Covering

Simplifying Boolean Equations

Example 6:

$$Y = (A + BC)(A + DE)$$

Apply T8' first when possible: $W + XZ = (W + X)(W + Z)$

Make: $X = BC$, $Z = DE$ and rewrite equation

$$Y = (A + X)(A + Z)$$

substitution ($X = BC$, $Z = DE$)

$$= A + XZ$$

T8': Distributivity

$$= A + BCDE$$

substitution

or

$$Y = AA + ADE + ABC + BCDE$$

T8: Distributivity

$$= A + ADE + ABC + BCDE$$

T3: Idempotency

$$= A + ADE + ABC + BCDE$$

$$= A + ABC + BCDE$$

T9': Covering

$$= A + BCDE$$

T9': Covering

This is called *multiplying out* an expression to get *sum-of-products (SOP)* form.

Multiplying Out: SOP Form

An expression is in simplified **sum-of-products (SOP)** form when all products contain literals only.

- SOP form: $Y = AB + BC' + DE$
- **NOT** SOP form: $Y = DF + E(A' + B)$
- SOP form: $Z = A + BC + DE'F$

Multiplying Out: SOP Form

Example:

$$Y = (A + C + D + E)(A + B)$$

Apply T8' first when possible: $W + XZ = (W + X)(W + Z)$

Multiplying Out: SOP Form

Example:

$$Y = (A + C + D + E)(A + B)$$

Apply T8' first when possible: $W+XZ = (W+X)(W+Z)$

Make: $X = (C+D+E)$, $Z = B$ and rewrite equation

$$Y = (A+X)(A+Z)$$

substitution ($X=(C+D+E)$, $Z=B$)

$$= A + XZ$$

T8': Distributivity

$$= A + (C+D+E)B$$

substitution

$$= A + BC + BD + BE$$

T8: Distributivity

or

$$Y = AA+AB+AC+BC+AD+BD+AE+BE$$

T8: Distributivity

$$= A+AB+AC+AD+AE+BC+BD+BE$$

T3: Idempotency

$$= A + BC + BD + BE$$

T9': Covering



Canonical SOP & POS Form

- SOP – sum-of-products

O	C	E	minterm
0	0	0	$\overline{O} \overline{C}$
0	1	0	$\overline{O} C$
1	0	1	$O \overline{C}$
1	1	0	$O C$

$$E = O\overline{C}$$

$$= \Sigma(m_2)$$

- POS – product-of-sums

O	C	E	maxterm
0	0	0	$O + C$
0	1	0	$O + \overline{C}$
1	0	1	$\overline{O} + C$
1	1	0	$\overline{O} + \overline{C}$

$$E = (O + C)(O + \overline{C})(\overline{O} + \overline{C})$$

$$= \Pi(M_0, M_1, M_3)$$

Factoring: POS Form

An expression is in simplified **product-of-sums (POS)** form when all sums contain literals only.

- POS form: $Y = (A+B)(C+D)(E'+F)$
- **NOT** POS form: $Y = (D+E)(F'+GH)$
- POS form: $Z = A(B+C)(D+E')$

Factoring: POS Form

Example 1:

$$Y = (A + B' CDE)$$

Apply T8' first when possible: $W + XZ = (W + X)(W + Z)$

Factoring: POS Form

Example 1:

$$Y = (A + B'CDE)$$

Apply T8' first when possible: $W+XZ = (W+X)(W+Z)$

Make: $X = B'C$, $Z = DE$ and rewrite equation

$$Y = (A+XZ)$$

substitution ($X=B'C$, $Z=DE$)

$$= (A+B'C)(A+DE)$$

T8': Distributivity

$$= (A+B')(A+C)(A+D)(A+E)$$

T8': Distributivity

Factoring: POS Form

Example 2:

$$Y = AB + C'DE + F$$

Apply T8' first when possible: $W+XZ = (W+X)(W+Z)$



Factoring: POS Form

Example 2:

$$Y = AB + C'DE + F$$

Apply T8' first when possible: $W+XZ = (W+X)(W+Z)$

Make: $W = AB$, $X = C'$, $Z = DE$ and rewrite equation

$$\begin{aligned} Y &= (W+XZ) + F && \text{substitution } W = AB, X = C', Z = DE \\ &= (W+X)(W+Z) + F && \text{T8': Distributivity} \\ &= (AB+C')(AB+DE)+F && \text{substitution} \\ &= (A+C')(B+C')(AB+D)(AB+E)+F && \text{T8': Distributivity} \\ &= (A+C')(B+C')(A+D)(B+D)(A+E)(B+E)+F && \text{T8': Distributivity} \\ &= (A+C'+F)(B+C'+F)(A+D+F)(B+D+F)(A+E+F)(B+E+F) && \text{T8': Distributivity} \end{aligned}$$

Boolean Thms of Several Vars: Duals

#	Theorem	Dual	Name
T6	$B \bullet C = C \bullet B$	$B + C = C + B$	Commutativity
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	$(B + C) + D = B + (C + D)$	Associativity
T8	$B \bullet (C + D) = (B \bullet C) + (B \bullet D)$	$B + (C \bullet D) = (B + C) (B + D)$	Distributivity
T9	$B \bullet (B + C) = B$	$B + (B \bullet C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	$(B + C) \bullet (B + \overline{C}) = B$	Combining
T11	$(B \bullet C) + (B \bullet \overline{D}) + (C \bullet D) = (B \bullet C) + (B \bullet \overline{D})$	$(B + C) \bullet (B + \overline{D}) \bullet (C + D) = (B + C) \bullet (B + \overline{D})$	Consensus

Axioms and theorems are useful for *simplifying* equations.

Simplification methods

- **Distributivity (T8, T8')**

$$B(C+D) = BC + BD$$

$$B + CD = (B + C)(B + D)$$

- **Covering (T9')**

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- **Duplication**

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- **A combination of Combining/Covering**

$$PA + \overline{A} = P + \overline{A}$$

DeMorgan's Theorem

Number	Theorem	Name
T12	$\overline{B_0 \bullet B_1 \bullet B_2 \dots} = \overline{B_0} + \overline{B_1} + \overline{B_2} \dots$	DeMorgan's Theorem

DeMorgan's Theorem

Number	Theorem	Name
T12	$\overline{B_0 \bullet B_1 \bullet B_2 \dots} = \overline{B_0} + \overline{B_1} + \overline{B_2} \dots$	DeMorgan's Theorem

The **complement of the product**
is the
sum of the complements

DeMorgan's Theorem: Dual

#	Theorem	Dual	Name
T12	$\overline{B_0 \bullet B_1 \bullet B_2 \dots} = \overline{B_0} + \overline{B_1} + \overline{B_2} \dots$	$\overline{B_0 + B_1 + B_2 \dots} = \overline{B_0} \bullet \overline{B_1} \bullet \overline{B_2} \dots$	DeMorgan's Theorem

DeMorgan's Theorem: Dual

#	Theorem	Dual	Name
T12	$\overline{B_0 \bullet B_1 \bullet B_2 \dots} = \overline{B_0 + B_1 + B_2 \dots}$	$\overline{B_0 + B_1 + B_2 \dots} = \overline{B_0 \bullet B_1 \bullet B_2 \dots}$	DeMorgan's Theorem

The complement of the product
is the
sum of the complements.

DeMorgan's Theorem: Dual

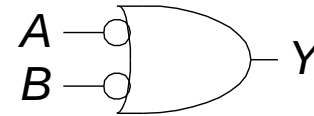
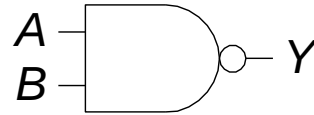
#	Theorem	Dual	Name
T12	$\overline{B_0 \bullet B_1 \bullet B_2 \dots} = \overline{B_0 + B_1 + B_2 \dots}$	$\overline{B_0 + B_1 + B_2 \dots} = \overline{B_0 \bullet B_1 \bullet B_2 \dots}$	DeMorgan's Theorem

The complement of the product
is the
sum of the complements.

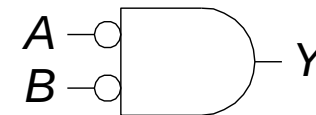
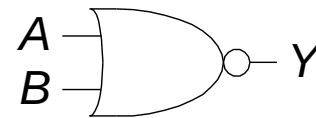
Dual: The complement of the sum
is the
product of the complements.

DeMorgan's Theorem

- $Y = \overline{AB} = \overline{A} + \overline{B}$



- $Y = \overline{A + B} = \overline{A} \cdot \overline{B}$



DeMorgan's Theorem Example 1

$$Y = \overline{(A+BD)C}$$

DeMorgan's Theorem Example 1

$$\begin{aligned} Y &= \overline{(\overline{A+BD})\overline{C}} \\ &= \overline{(\overline{A+BD})} + \overline{\overline{C}} \\ &= (\overline{A} \bullet \overline{(\overline{BD})}) + C \\ &= (\overline{A} \bullet (BD)) + C \\ &= \overline{A}BD + C \end{aligned}$$

DeMorgan's Theorem Example 2

$$Y = \overline{(\overline{ACE} + \overline{D})} + B$$

DeMorgan's Theorem Example 2

$$\begin{aligned} Y &= \overline{\overline{ACE + D}} + \overline{B} \\ &= \overline{\overline{ACE + D}} \bullet \overline{B} \\ &= \overline{\overline{ACE} \bullet \overline{D}} \bullet \overline{B} \\ &= ((\overline{\overline{AC}} + \overline{E}) \bullet D) \bullet \overline{B} \\ &= ((AC + \overline{E}) \bullet D) \bullet \overline{B} \\ &= (ACD + D\overline{E}) \bullet \overline{B} \\ &= \overline{A}\overline{B}CD + \overline{B}D\overline{E} \end{aligned}$$

Canonical SOP & POS Form Revisited

- SOP – sum-of-products

O	C	E	minterm
0	0	0	$\overline{O} \overline{C}$
0	1	0	$\overline{O} C$
1	0	1	$O \overline{C}$
1	1	0	$O C$

How do we implement this logic function with gates?

$$E = O\overline{C}$$

$$= \Sigma(m_2)$$

- POS – product-of-sums

O	C	E	maxterm
0	0	0	$O + C$
0	1	0	$O + \overline{C}$
1	0	1	$\overline{O} + C$
1	1	0	$\overline{O} + \overline{C}$

$$E = (O + C)(O + \overline{C})(\overline{O} + \overline{C})$$

$$= \Pi(M_0, M_1, M_3)$$

Canonical POS Expansion

- “Add” literal/complement terms to reverse simplification ($\rightarrow$ expand literal)
- Example
 - $Y = C$
 - $Y = C + A\bar{A}$
 - $Y = (C + A) \cdot (C + \bar{A})$
 - $Y = [(C + A) + B\bar{B}](C + \bar{A})$
 - $Y = [(C + A + B)(C + A + \bar{B})](C + \bar{A})$
 - ...

Reminder

Midterm 1: Wednesday, Sep. 30

- 1 hour and 15 minutes + 15 mins for scan and upload on Canvas
- Chap 1 – 2.6
- Closed book, closed notes
- No calculator
- Boolean Theorems & Axioms document will be attached to the exam for your convenience

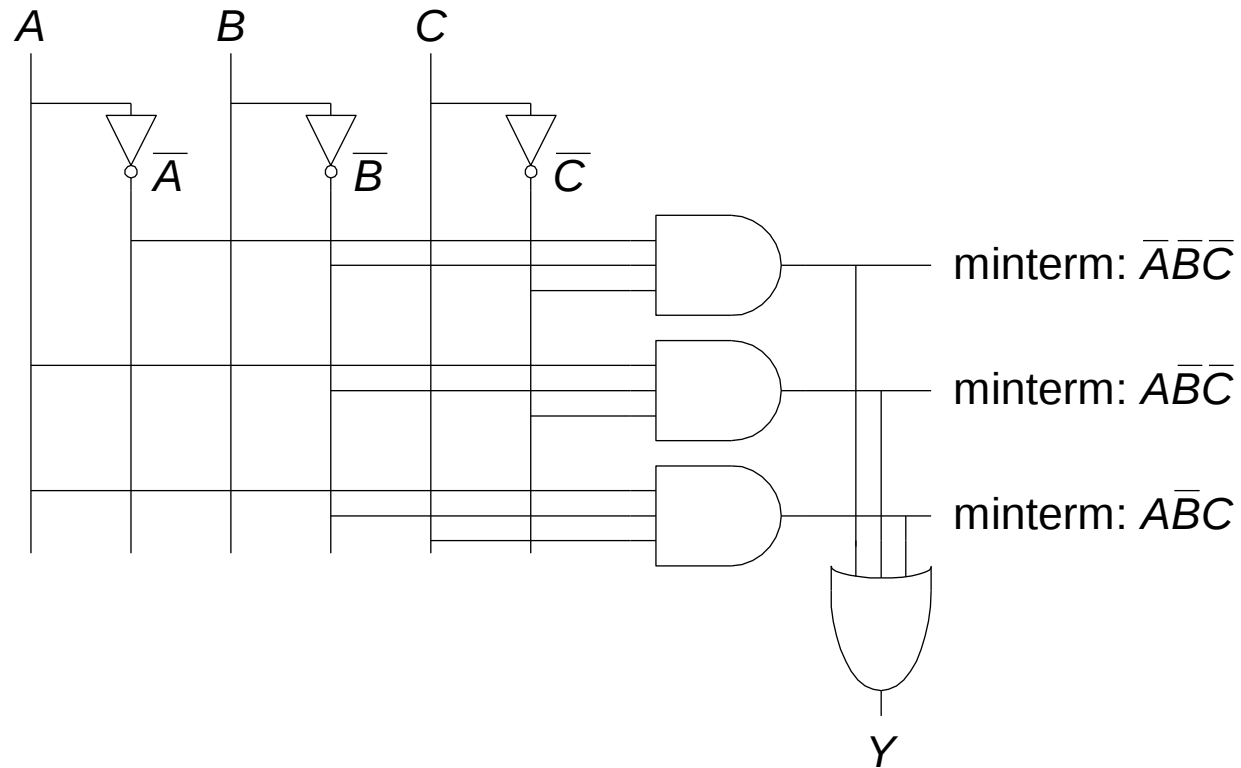
Chapter 2.4

From Logic to Gates



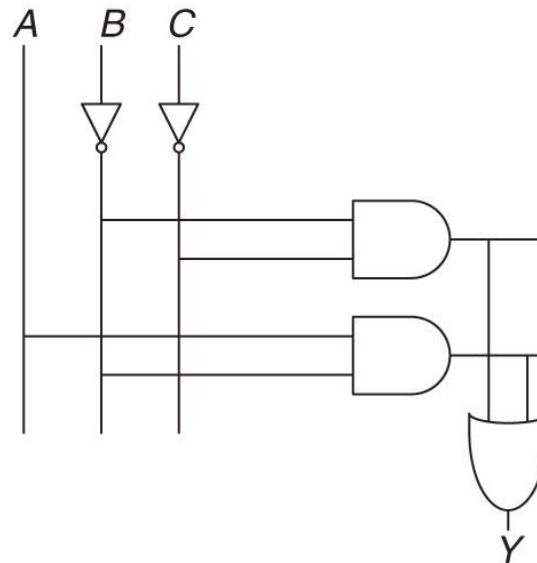
From Logic to Gates

- Two-level logic: ANDs followed by ORs
- Example: $Y = \bar{A}\bar{B}\bar{C} + A\bar{B}\bar{C} + A\bar{B}C$



Circuit Schematics Rules

- Inputs on the left (or top)
- Outputs on right (or bottom)
- Gates flow from left to right
- Straight wires are best

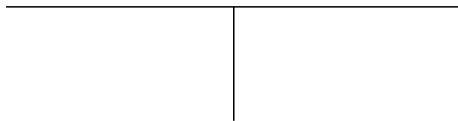


$$Y = \bar{B}\bar{C} + A\bar{B}$$

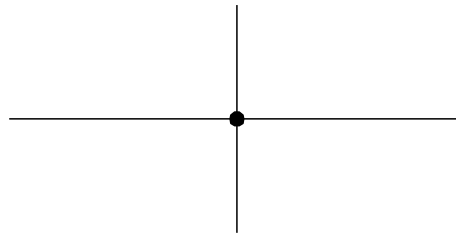
Circuit Schematic Rules (cont.)

- Wires always connect at a T junction
- A dot where wires cross indicates a connection between the wires
- Wires crossing *without* a dot make no connection

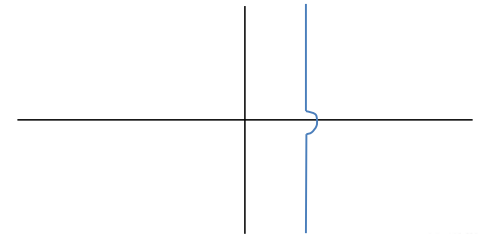
wires connect
at a T junction



wires connect
at a dot



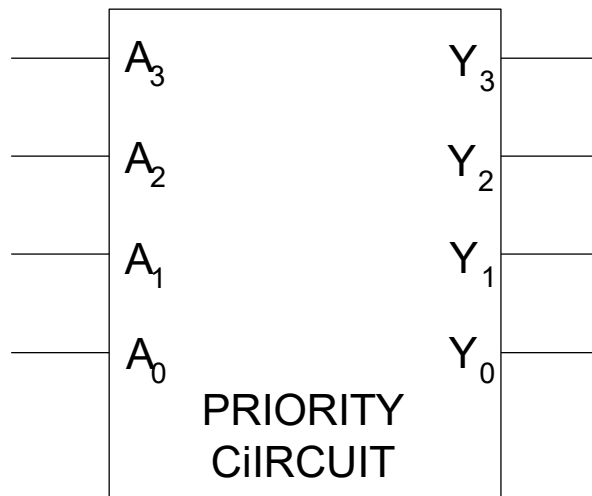
wires crossing
without a dot do
not connect



Multiple-Output Circuits

- Example: Priority Circuit**

Output asserted
corresponding to
most significant
TRUE input

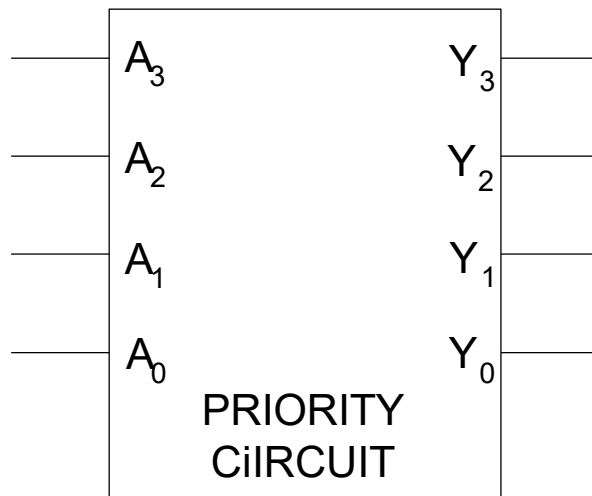


A_3	A_2	A_1	A_0	Y_3	Y_2	Y_1	Y_0
0	0	0	0				
0	0	0	1				1
0	0	1	0			1	
0	0	1	1			1	
0	1	0	0		1		
0	1	0	1		1		
0	1	1	0		1		
0	1	1	1		1		
1	0	0	0	1			
1	0	0	1	1			
1	0	1	0	1			
1	0	1	1	1			
1	1	0	0	1			
1	1	0	1	1			
1	1	1	0	1			
1	1	1	0	1			
1	1	1	1	1			
1	1	1	1	1			

Multiple-Output Circuits

- Example: Priority Circuit**

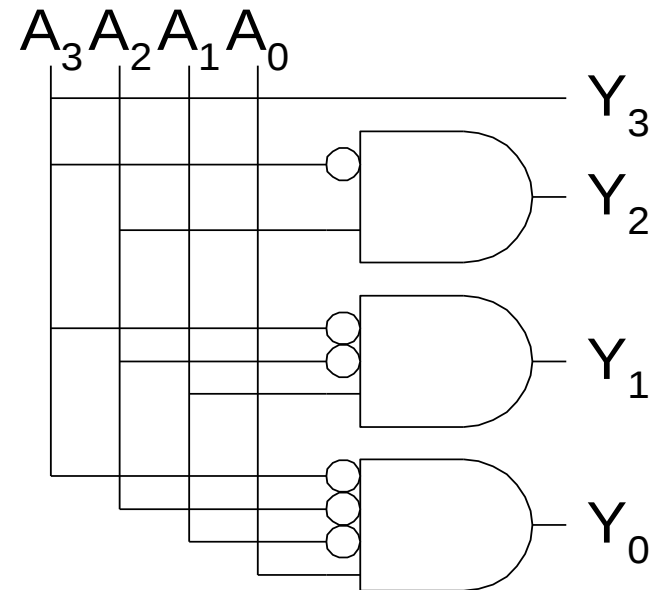
Output asserted
corresponding to
most significant
TRUE input



A_3	A_2	A_1	A_0	Y_3	Y_2	Y_1	Y_0
0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1
0	0	1	0	0	0	1	0
0	0	1	1	0	0	1	0
0	1	0	0	0	1	0	0
0	1	0	1	0	1	0	0
0	1	1	0	0	1	0	0
0	1	1	1	0	1	0	0
1	0	0	0	1	0	0	0
1	0	0	1	1	0	0	0
1	0	1	0	1	0	0	0
1	0	1	1	1	0	0	0
1	1	0	0	1	0	0	0
1	1	0	1	1	0	0	0
1	1	1	0	1	0	0	0
1	1	1	1	1	0	0	0
1	1	1	1	1	0	0	0

Priority Circuit Hardware

A_3	A_2	A_1	A_0	Y_3	Y_2	Y_1	Y_0
0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1
0	0	1	0	0	0	1	0
0	0	1	1	0	0	1	0
0	1	0	0	0	1	0	0
0	1	0	1	0	1	0	0
0	1	1	0	0	1	0	0
0	1	1	1	0	1	0	0
1	0	0	0	1	0	0	0
1	0	0	1	1	0	0	0
1	0	1	0	1	0	0	0
1	0	1	1	1	0	0	0
1	1	0	0	1	0	0	0
1	1	0	1	1	0	0	0
1	1	1	0	1	0	0	0
1	1	1	1	1	0	0	0



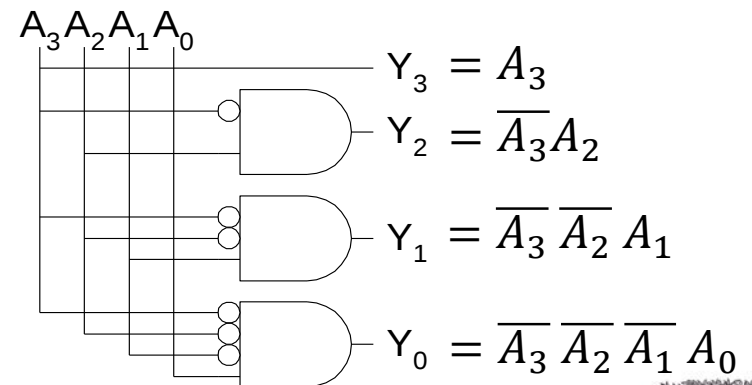
Don't Cares

- Simplify truth table by ignoring entries

A_3	A_2	A_1	A_0	Y_3	Y_2	Y_1	Y_0
0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1
0	0	1	0	0	0	1	0
0	0	1	1	0	0	1	0
0	1	0	0	0	1	0	0
0	1	0	1	0	1	0	0
0	1	1	0	0	1	0	0
0	1	1	1	0	1	0	0
1	0	0	0	1	0	0	0
1	0	0	1	1	0	0	0
1	0	1	0	1	0	0	0
1	0	1	1	1	0	0	0
1	1	0	0	1	0	0	0
1	1	0	1	1	0	0	0
1	1	1	0	1	0	0	0
1	1	1	1	1	0	0	0

A_3	A_2	A_1	A_0	Y_3	Y_2	Y_1	Y_0
0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1
0	0	1	X	0	0	1	0
0	1	X	X	0	1	0	0
1	X	X	X	1	0	0	0

Much easier to read off Boolean equations



Chapter 2.5

Multilevel Combinational Logic



Multilevel Hardware Simplification

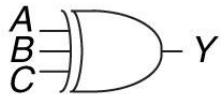
- Two-level logic: One level-ANDs, one-level OR
 - Can have high hardware requirements
 - Multiple inputs to AND and OR gates
- Multilevel design can simplify schematic

Multilevel Hardware Simplification

- 3-input XOR example (odd parity)

- $$Y = \bar{A}\bar{B}C + \bar{A}B\bar{C} + A\bar{B}\bar{C} + ABC$$

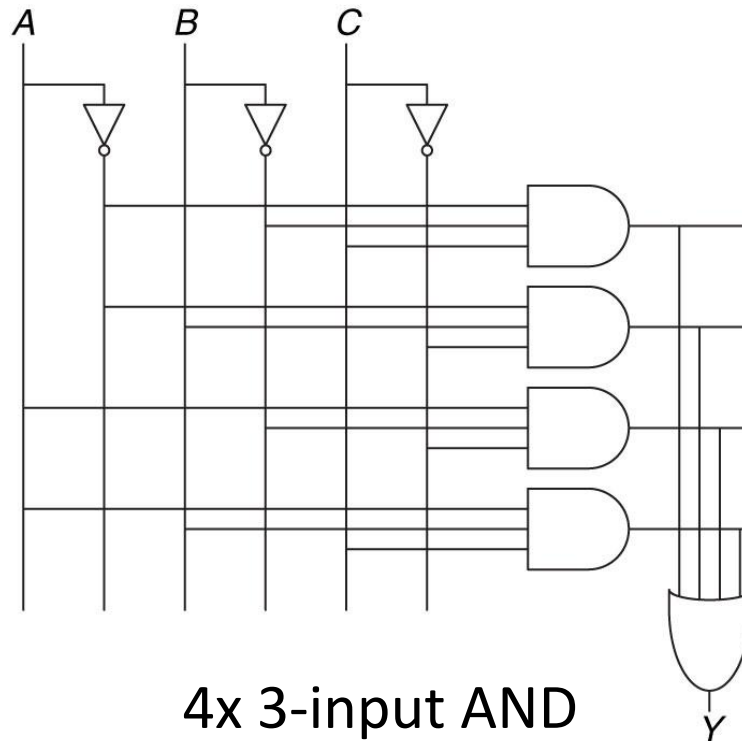
XOR3



$$Y = A \oplus B \oplus C$$

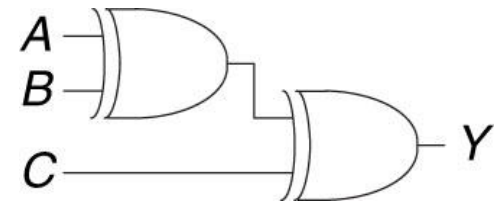
A	B	C	Y
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

(a)



(b)

4x 3-input AND
1x 4-input OR

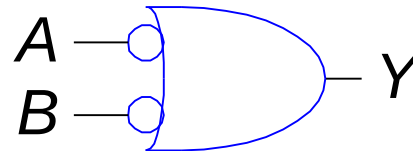
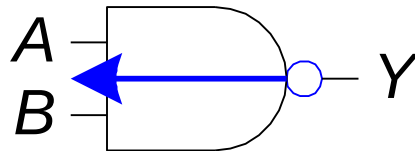


2x 2-input XOR

Bubble Pushing

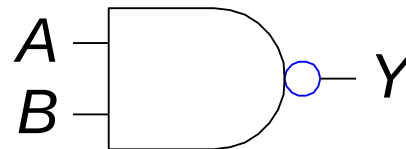
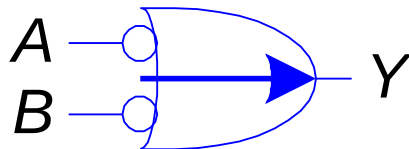
- **Backward:**

- Body changes
- Adds bubbles to inputs



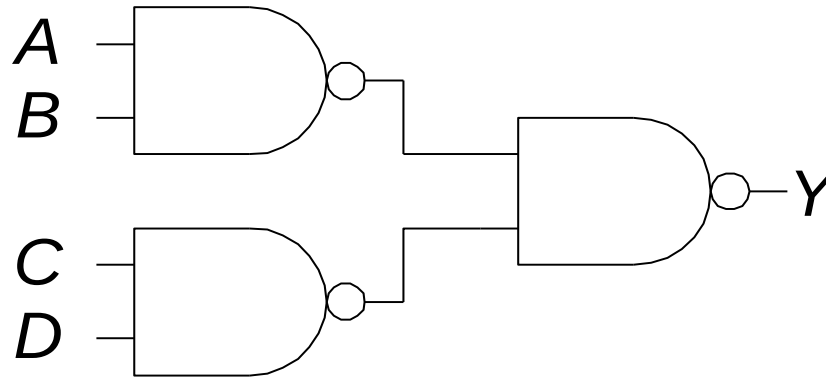
- **Forward:**

- Body changes
- Adds bubble to output



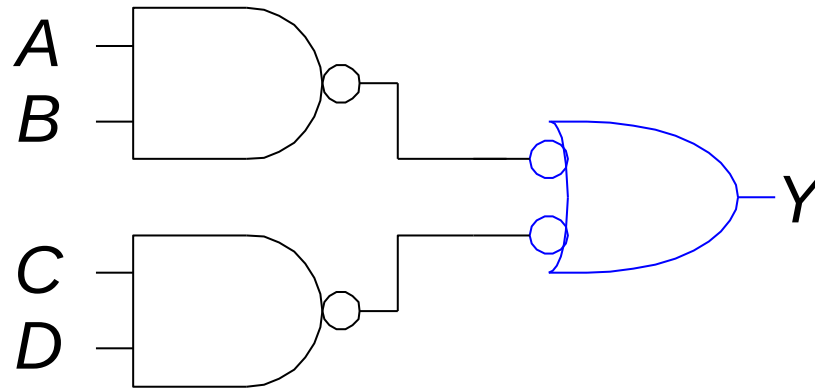
Bubble Pushing

- What is the Boolean expression for this circuit?



Bubble Pushing

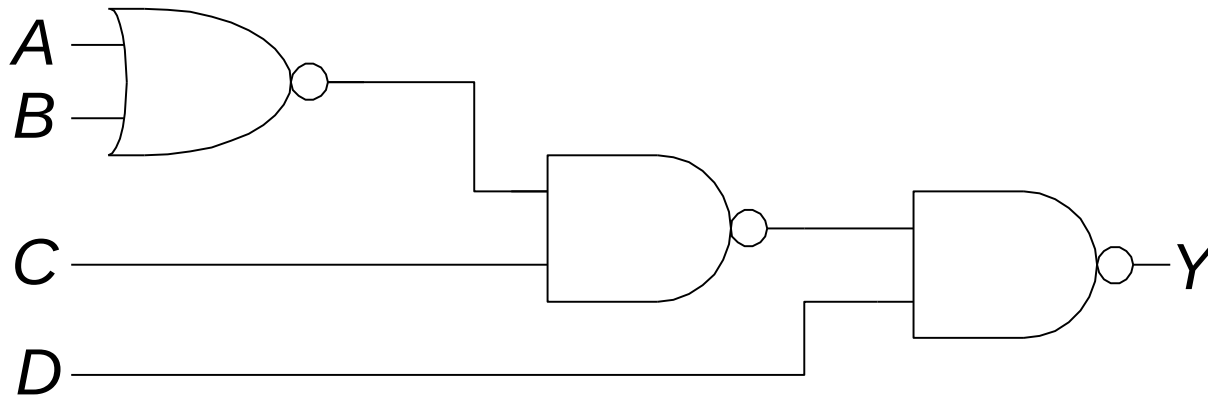
- What is the Boolean expression for this circuit?



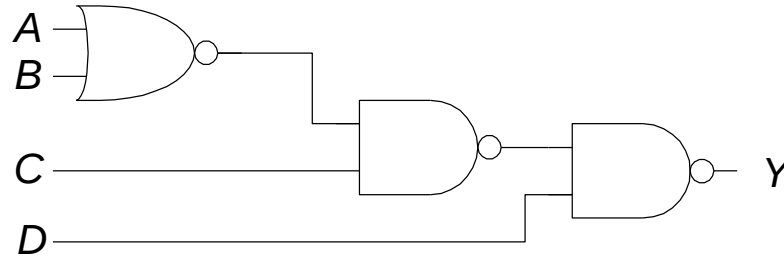
$$Y = AB + CD$$

Bubble Pushing Rules

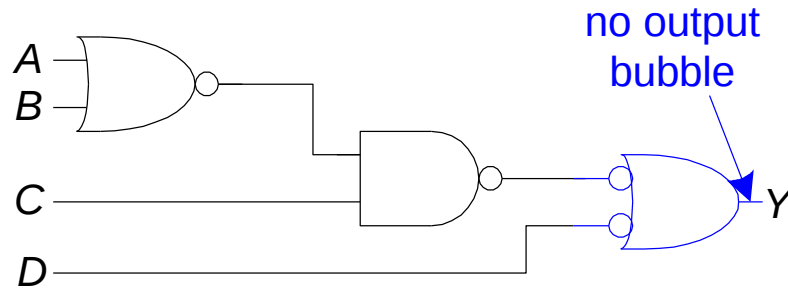
- Begin at output, then work toward inputs
- Push bubbles on final output back
- Draw gates in a form so bubbles cancel



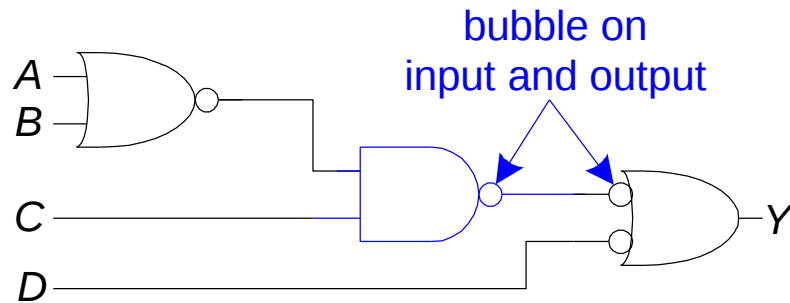
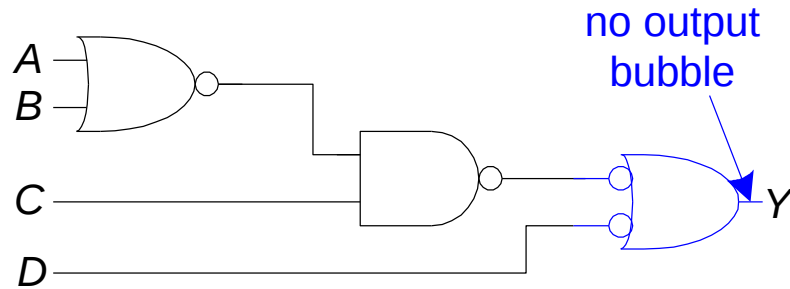
Bubble Pushing Example



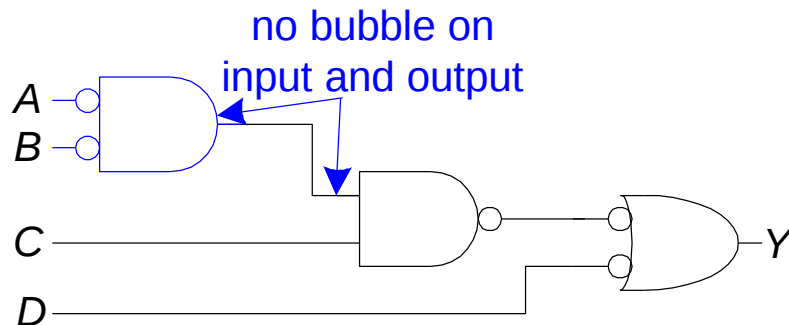
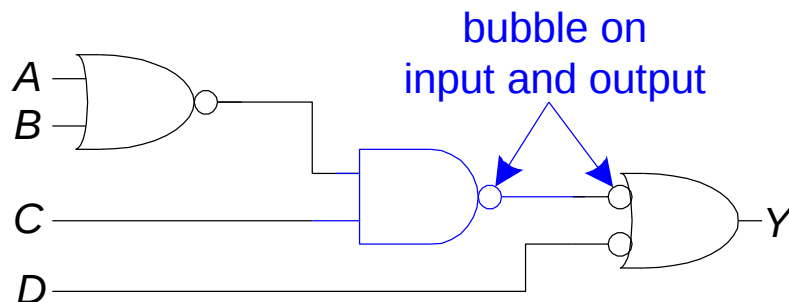
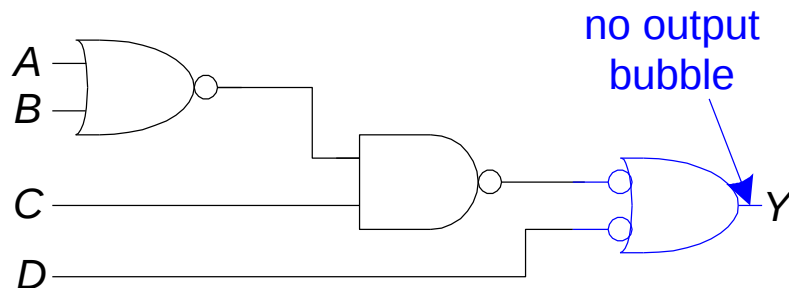
Bubble Pushing Example



Bubble Pushing Example



Bubble Pushing Example



$$Y = \overline{A} \overline{B} C + \overline{D}$$

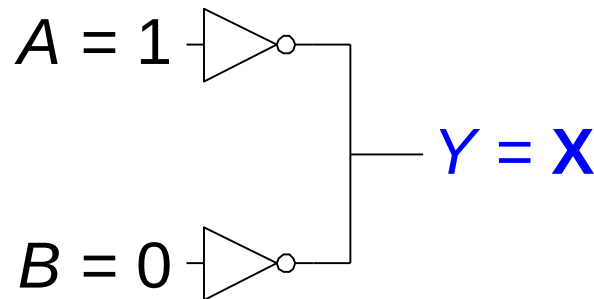
Chapter 2.6

X's and Z's, Oh My



Contention: X

- Contention: circuit tries to drive output to 1 **and** 0
 - Actual value somewhere in between
 - Could be 0, 1, or in forbidden zone
 - Might change with voltage, temperature, time, noise
 - Often causes excessive power dissipation

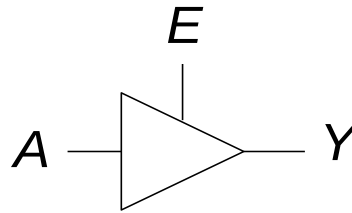


- **Warnings:**
 - Contention usually indicates a **bug**.
 - **X** is used for “don’t care” and contention - look at the context to tell them apart

Floating: Z

- Floating, high impedance, open, high Z
- Floating output might be 0, 1, or somewhere in between
 - A voltmeter won't indicate whether a node is floating

Tristate Buffer

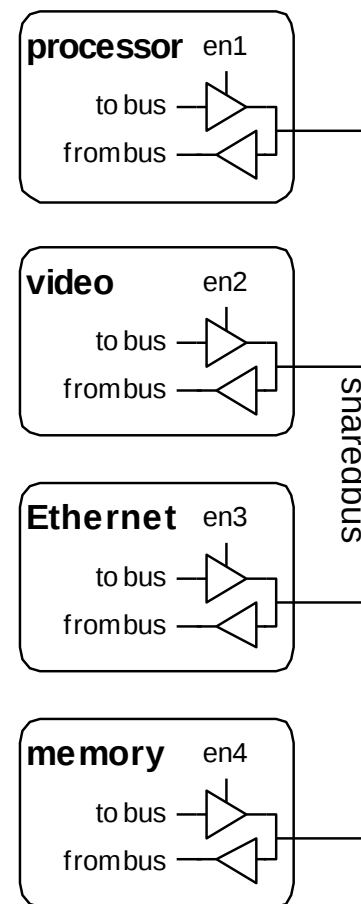


E	A	Y
0	0	Z
0	1	Z
1	0	0
1	1	1

Note: tristate buffer has an enable bit (E) to turn on the gate

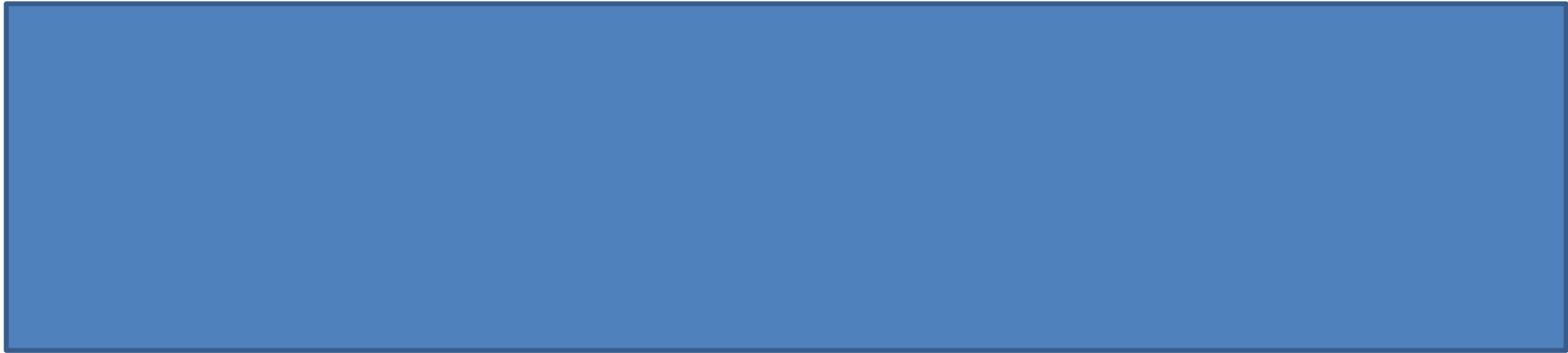
Tristate Busses

- Floating nodes are used in tristate busses
 - Many different drivers
 - Exactly one is active at once



Chapter 2.7

Karnaugh Maps



Karnaugh Maps (K-Maps)

- Boolean expressions can be minimized by combining terms
 - $PA + \overline{PA} = P$
- K-maps minimize equations graphically
 - Put terms to combine close to one another

A	B	C	Y
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

		AB			
		00	01	11	10
C	0	1	0	0	0
	1	1	0	0	0

		AB			
		00	01	11	10
C	0	$\overline{A}\overline{B}\overline{C}$	$\overline{A}B\overline{C}$	$AB\overline{C}$	$A\overline{B}\overline{C}$
	1	$\overline{A}\overline{B}C$	$\overline{A}BC$	ABC	$A\overline{B}C$

$$Y = \overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}C = \overline{A}\overline{B}(C + \overline{C})$$



K-Map

- Circle 1's in adjacent squares
- In Boolean expression, include only literals whose true and complement form are **not** in the circle

A	B	C	Y
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

		AB			
C	Y	00	01	11	10
	0	1	0	0	0
1	1	1	0	0	0

$$Y = \bar{A}\bar{B}$$

C not included because both C and $\bar{C}$ included in circle



3-Input K-Map

		AB			
		00	01	11	10
C	0	$\bar{A}\bar{B}\bar{C}$	$\bar{A}B\bar{C}$	$AB\bar{C}$	$A\bar{B}\bar{C}$
	1	$\bar{A}\bar{B}C$	$\bar{A}BC$	ABC	$A\bar{B}C$

Truth Table

A	B	C	Y
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

K-Map

		AB			
		00	01	11	10
C	0				
	1				

K-Map Definitions

- **Complement:** variable with a bar over it
 $\bar{A}, \bar{B}, \bar{C}$
- **Literal:** variable or its complement
 $\bar{A}, A, \bar{B}, B, C, \bar{C}$
- **Implicant:** product of literals
 $\bar{A}\bar{B}C, \bar{A}C, BC$
- **Prime implicant:** implicant corresponding to the largest circle in a K-map

K-Map Rules

- Every **1 must be circled** at least once
- Each circle must span a **power of 2** (i.e. 1, 2, 4) squares in each direction
- Each **circle** must be **as large** as possible
- A circle may **wrap around the edges**
- A “don't care” (**X**) is **circled only if it helps** minimize the equation

4-Input K-Map

<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>Y</i>
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	1
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

<i>Y</i>		<i>AB</i>			
<i>CD</i>		00	01	11	10
	00				
	01				
	11				
	10				

4-Input K-Map

A	B	C	D	Y
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	1
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

Y CD \ AB				
	00	01	11	10
00	1	0	0	1
01	0	1	0	1
11	1	1	0	0
10	1	1	0	1

4-Input K-Map

A	B	C	D	Y
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	1
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

		AB			
Y	CD	00	01	11	10
	00	1	0	0	1
	01	0	1	0	1
	11	1	1	0	0
	10	1	1	0	1

$$Y = \bar{A}\bar{C} + \bar{A}BD + A\bar{B}\bar{C} + \bar{B}\bar{D}$$

K-Maps with Don't Cares

A	B	C	D	Y
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	X
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	X
1	0	1	1	X
1	1	0	0	X
1	1	0	1	X
1	1	1	0	X
1	1	1	1	X

		AB			
Y	CD	00	01	11	10
	00				
	01				
	11				
	10				

K-Maps with Don't Cares

A	B	C	D	Y
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	X
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	X
1	0	1	1	X
1	1	0	0	X
1	1	0	1	X
1	1	1	0	X
1	1	1	1	X

		AB			
Y	CD	00	01	11	10
	00	1	0	X	1
	01	0	X	X	1
	11	1	1	X	X
	10	1	1	X	X

K-Maps with Don't Cares

A	B	C	D	Y
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	X
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	X
1	0	1	1	X
1	1	0	0	X
1	1	0	1	X
1	1	1	0	X
1	1	1	1	X

		AB			
Y	CD	00	01	11	10
		00	01	11	10
	00	1	0	X	1
	01	0	X	X	1
	11	1	1	X	X
	10	1	1	X	X

$$Y = A + \bar{B}\bar{D} + C$$

4-Input K-Map: POS & SOP Form

A	B	C	D	Y
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	1
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

		AB			
Y	CD	00	01	11	10
	00	1	0	0	1
	01	0	1	0	1
	11	1	1	0	0
	10	1	1	0	1

4-Input K-Map: POS Form

A	B	C	D	Y
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	1
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

		AB			
Y	CD	00	01	11	10
		00	01	11	10
	00	1	0	0	1
	01	0	1	0	1
	11	1	1	0	0
	10	1	1	0	1

$$Y = \bar{A}C + \bar{A}BD + A\bar{B}\bar{C} + \bar{B}\bar{D}$$

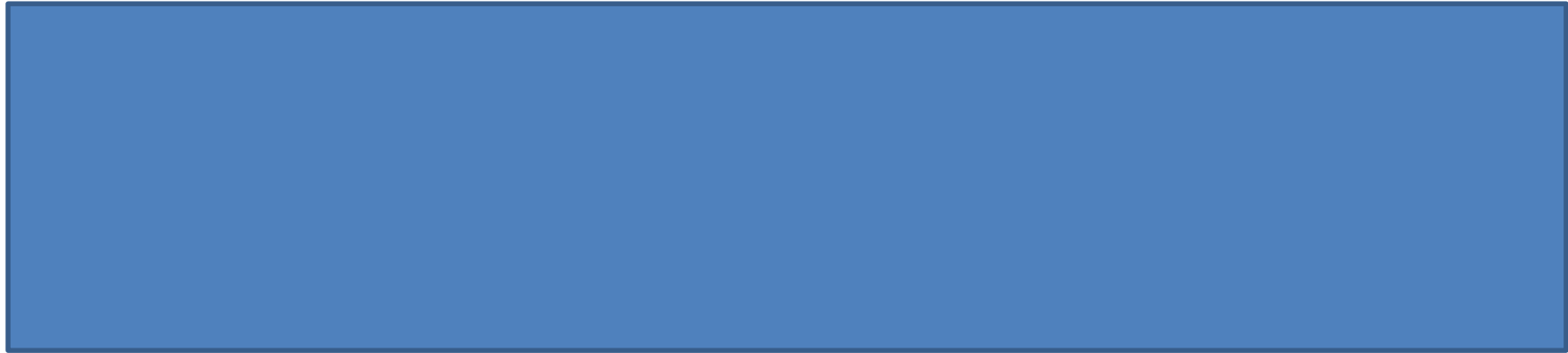
4-Input K-Map: POS Form

A	B	C	D	Y
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	1
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

		AB			
		00	01	11	10
CD	00	1	0	0	1
	01	0	1	0	1
	11	1	1	0	0
	10	1	1	0	1

Chapter 2.8

Combinational Building Blocks



Combinational Building Blocks

- Multiplexers
- Decoders

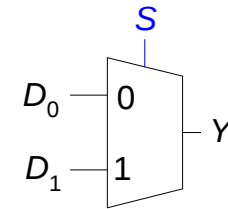
Multiplexer (Mux)

- Selects between one of N inputs to connect to output
 - $\log_2 N$ -bit required to select input – control input S

- Example:

2:1 Mux (2 inputs to 1 output)

- $N = 2$
- $\log_2 2 = 1$ control bit required



S	D_1	D_0	Y	S	Y
0	0	0	0	0	D_0
0	0	1	1	1	D_1
0	1	0	0		
0	1	1	1		
1	0	0	0		
1	0	1	0		
1	1	0	1		
1	1	1	1		

Multiplexer Implementations

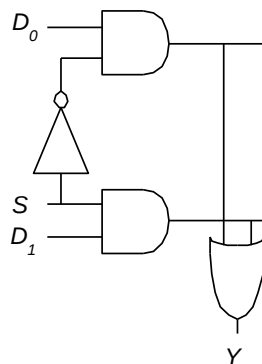
- Logic gates

- Sum-of-products form

S	D ₁	D ₀	Y
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

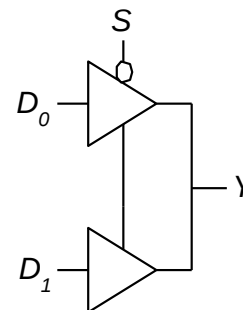
Y	S	D ₀ D ₁			
		00	01	11	10
	0	0	0	1	1
	1	0	1	1	0

$$Y = D_0 \bar{S} + D_1 S$$



- Tristates

- For an N-input mux, use N tristates
- Turn on exactly one to select the appropriate input

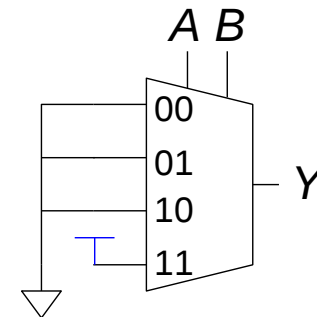


Logic using Multiplexers

- Using the mux as a lookup table
 - Zero outputs tied to GND
 - One output tied to VDD

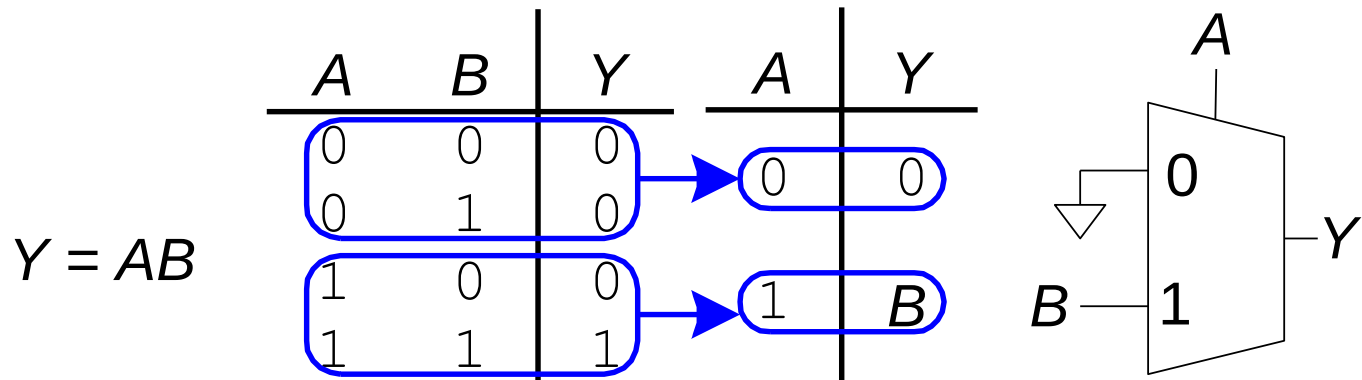
A	B	Y
0	0	0
0	1	0
1	0	0
1	1	1

$$Y = AB$$



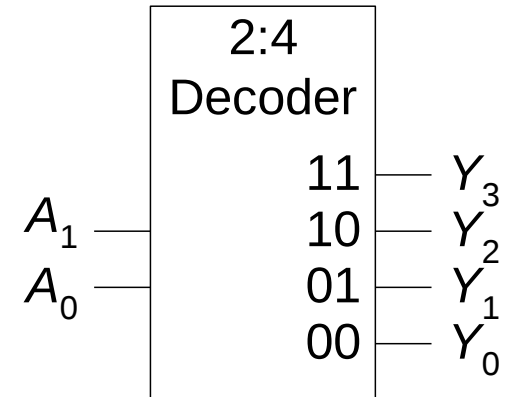
Logic using Multiplexers

- Reducing the size of the mux



Decoders

- N inputs, 2^N outputs
- One-hot outputs: only one output HIGH at once



- Example

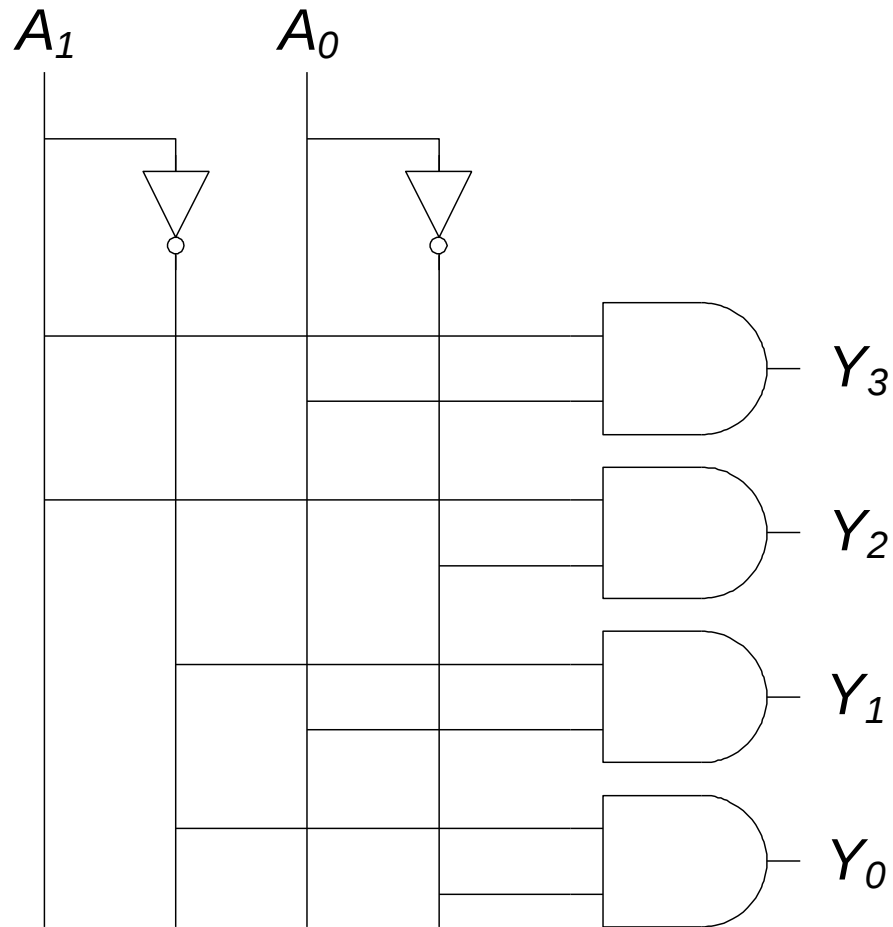
2:4 Decoder (2 inputs to 4 outputs)

- A_i decimal value selects the corresponding output

A_1	A_0	Y_3	Y_2	Y_1	Y_0
0	0	0	0	0	1
0	1	0	0	1	0
1	0	0	1	0	0
1	1	1	0	0	0

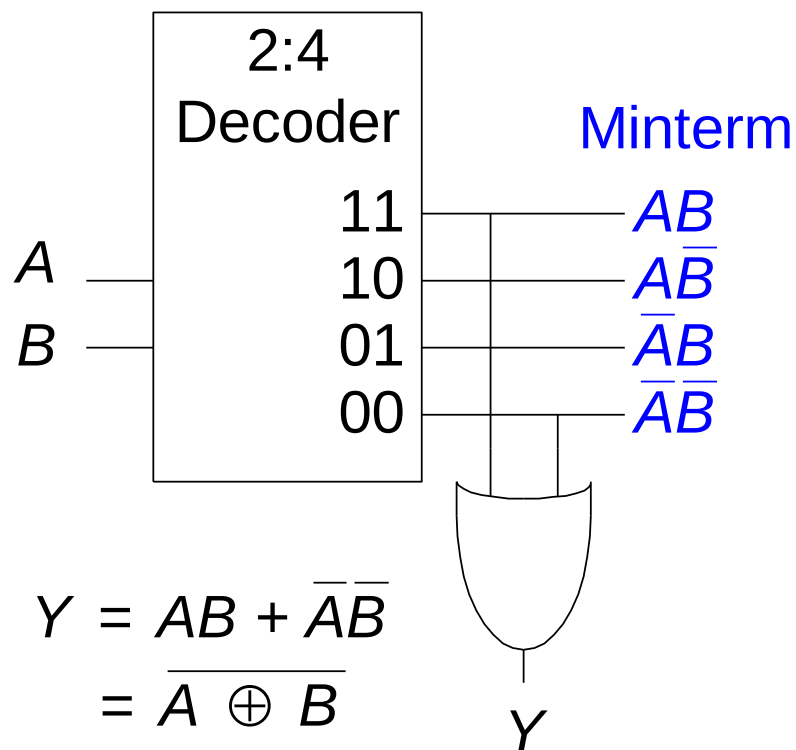


Decoder Implementation



Logic Using Decoders

- OR minterms



XNOR function

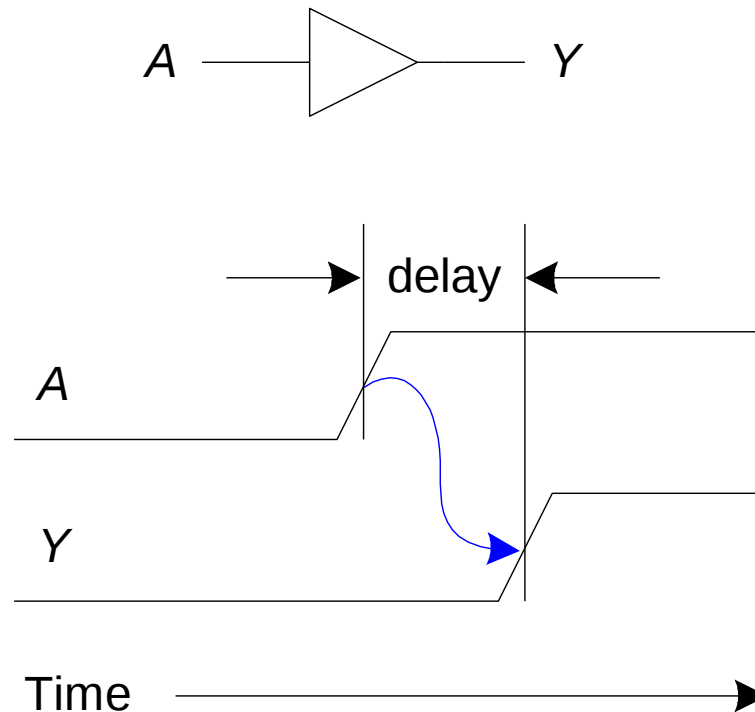
Chapter 2.9

Timing



Timing

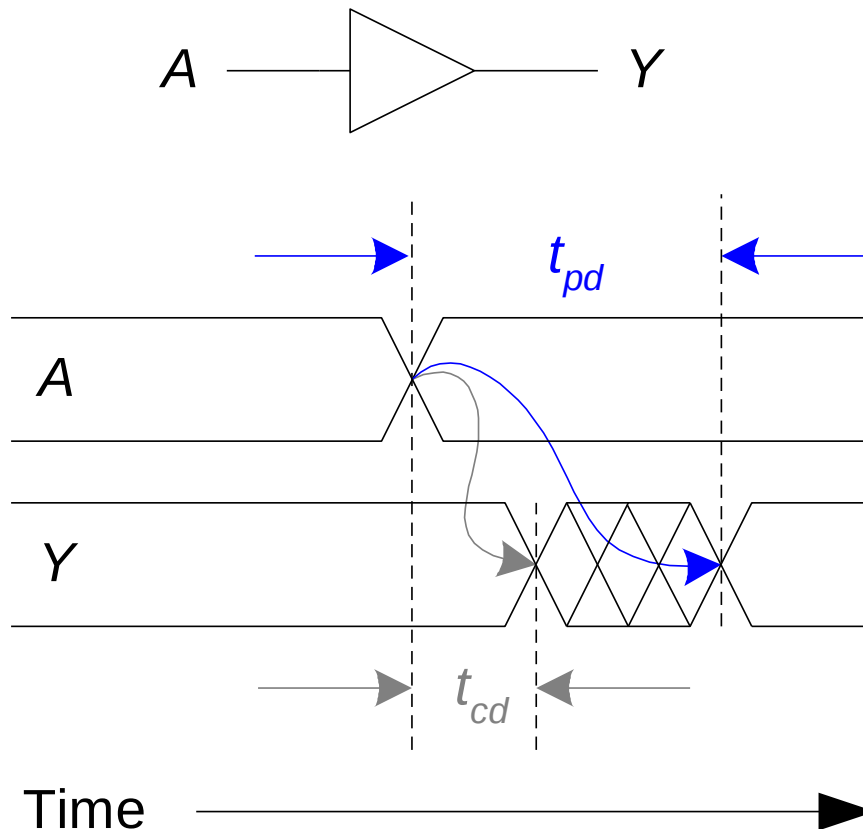
- Delay between input change and output changing



- How to build fast circuits?

Propagation & Contamination Delay

- **Propagation delay:** t_{pd} = max delay from input to final output
- **Contamination delay:** t_{cd} = min delay from input to initial output change



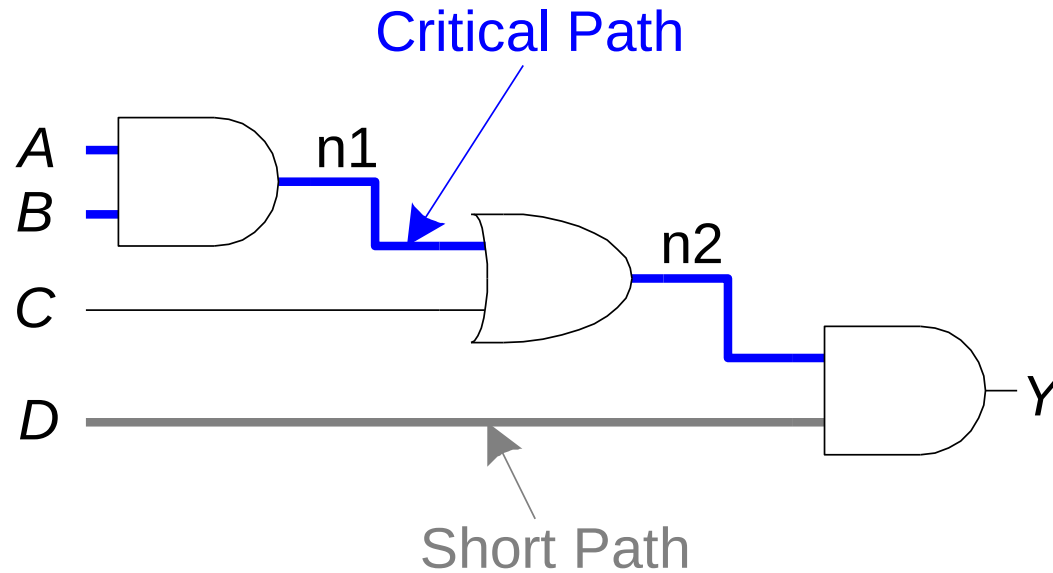
Note: Timing diagram shows a signal with a high and low and transition time as an 'X'.

Cross hatch indicates unknown/changing values

Propagation & Contamination Delay

- Delay is caused by
 - Capacitance and resistance in a circuit
 - Speed of light limitation
- Reasons why t_{pd} and t_{cd} may be different:
 - Different rising and falling delays
 - Multiple inputs and outputs, some of which are faster than others
 - Circuits slow down when hot and speed up when cold

Critical (Long) & Short Paths



Critical (Long) Path: $t_{pd} = 2t_{pd_AND} + t_{pd_OR}$

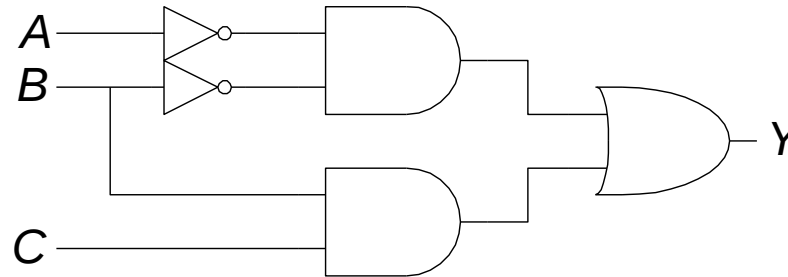
Short Path: $t_{cd} = t_{cd_AND}$

Glitches

- When a single input change causes an output to change multiple times

Glitch Example

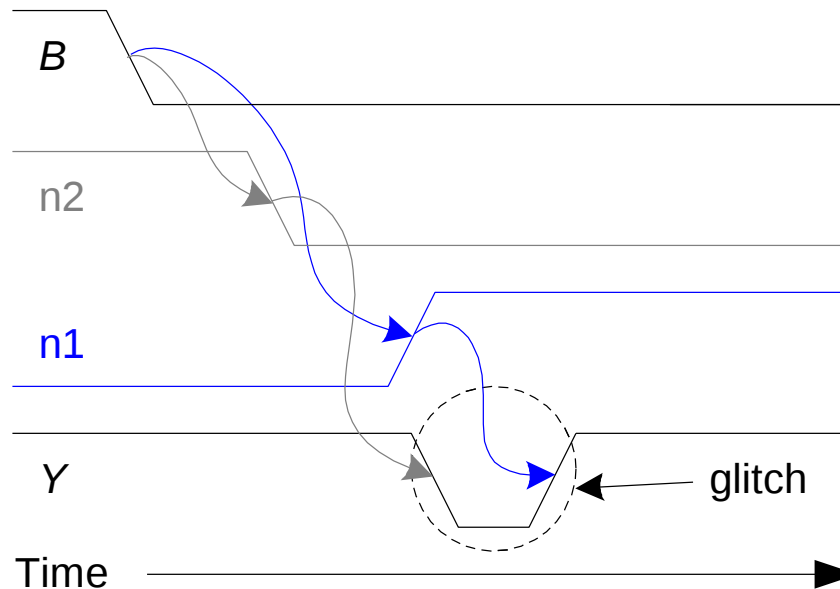
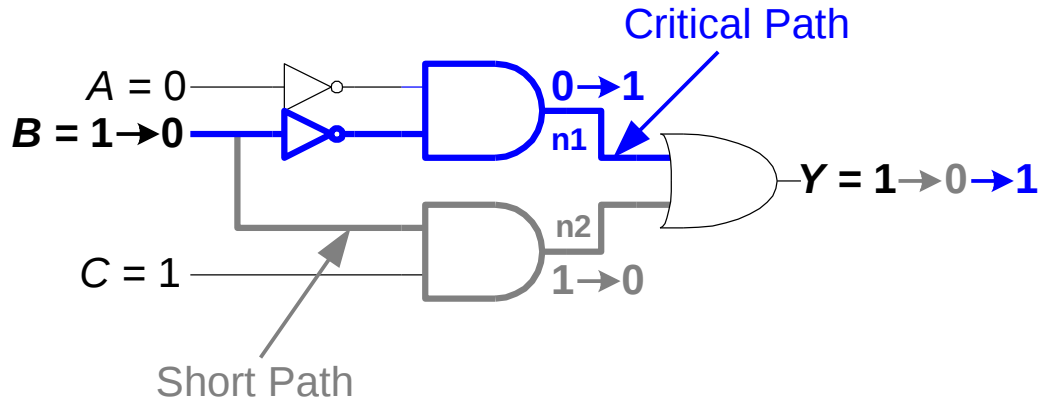
- What happens when $A = 0$, $C = 1$, B falls?



		AB			
		00	01	11	10
C	0	1	0	0	0
	1	1	1	1	0

$$Y = \bar{A}\bar{B} + BC$$

Glitch Example (cont.)



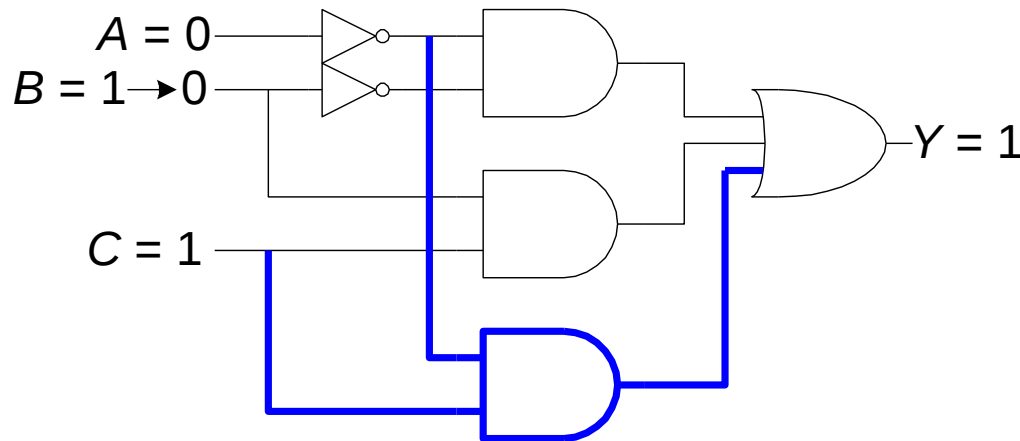
Note: $n1$ is slower than $n2$ because of the extra inverter for B to go through

Fixing the Glitch

Y C	AB			
	00	01	11	10
0	1	0	0	0
1	1	1	1	0

Consensus term $\bar{A}C$

$$Y = \bar{A}\bar{B} + BC + \bar{A}C$$



Why Understand Glitches?

- Glitches shouldn't cause problems because of **synchronous design** conventions (see Chapter 3)
- It's important to **recognize** a glitch: in simulations or on oscilloscope
- Can't get rid of all glitches – simultaneous transitions on multiple inputs can also cause glitches