COMBINATIONAL LOGIC DESIGN

Administrative Notes

- Note: New homework instructions starting with HW03
- Homework is due at the beginning of class
- Homework must be organized, legible (messy is not), and stapled to be graded

COMBINATIONAL LOGIC DESIGN

Some Definitions

- Complement: variable with a bar over it

  $\overline{A}, \overline{B}, \overline{C}$
- Literal: variable or its complement

  $A, \overline{A}, B, \overline{B}, C, \overline{C}$
- Implicant: product of literals

  $AB\overline{C}, \overline{A}C, BC$
- Minterm: product that includes all input variables

  $AB\overline{C}, A\overline{B}\overline{C}, ABC$
- Maxterm: sum that includes all input variables

  $(A+\overline{B}+C), (\overline{A}+B+\overline{C}), (\overline{A}+\overline{B}+C)$

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Canonical Sum-of-Products (SOP) Form

- All equations can be written in SOP form
- Each row has a **minterm**
- A minterm is a product (AND) of literals
- Each minterm is TRUE for that row (and only that row)

A	B	Y	minterm	minterm name
0	0	0	$\overline{A}\ \overline{B}$	m_0
0	1	1	$\overline{A}\ B$	m_1
1	0	0	$A\ \overline{B}$	m_2
1	1	1	$A\ B$	m_3

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- Form function by ORing minterms where the output is TRUE

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1	0	0	$A\ \overline{B}$	m_2
1	1	1	$A\ B$	m_3

$Y = F(A, B) =$

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- All equations can be written in SOP form
- Each row has a **minterm**
- A minterm is a product (AND) of literals
- Each minterm is TRUE for that row (and only that row)
- Form function by ORing minterms where the output is TRUE
- Thus, a sum (OR) of products (AND terms)

A	*B*	*Y*	minterm	minterm name
0	0	0	$\overline{A}\ \overline{B}$	m_0
0	1	1	$\overline{A}\ B$	m_1
1	0	0	$A\ \overline{B}$	m_2
1	1	1	$A\ B$	m_3

$$Y = F(A, B) = \overline{A}B + AB = \Sigma(m_1, m_3)$$

SOP Example

- Steps:
- Find minterms that result in Y=1
- Sum “TRUE” minterms

A	B	Y
0	0	1
0	1	1
1	0	0
1	1	0

$$Y = \mathrm{F}(A, B) =$$

Aside: Precedence

- AND has precedence over OR
- In other words:
  - AND is performed **before** OR
- Example:
  - $Y = \bar{A} \cdot B + A \cdot B$
  - Equivalent to:
  - $Y = (\bar{A}B) + (AB)$

Canonical Product-of-Sums (POS) Form

- All Boolean equations can be written in POS form
- Each row has a **maxterm**
- A maxterm is a sum (OR) of literals
- Each maxterm is FALSE for that row (and only that row)

A	*B*	*Y*	maxterm	maxterm name
0	0	0	$A + B$	M_0
0	1	1	$A + \overline{B}$	M_1
1	0	0	$\overline{A} + B$	M_2
1	1	1	$\overline{A} + \overline{B}$	M_3

Canonical Product-of-Sums (POS) Form

- All Boolean equations can be written in POS form
- Each row has a **maxterm**
- A maxterm is a sum (OR) of literals
- Each maxterm is FALSE for that row (and only that row)
- Form function by ANDing the maxterms for which the output is FALSE
- Thus, a product (AND) of sums (OR terms)

A	**B**	**Y**	**maxterm**	**maxterm name**
0	0	0	$A + B$	M_0
0	1	1	$A + \overline{B}$	M_1
1	0	0	$\overline{A} + B$	M_2
1	1	1	$\overline{A} + \overline{B}$	M_3

$$Y = M_0 \cdot M_2 = (A + B) \cdot (\bar{A} + B)$$

SOP and POS Comparison

- Sum of Products (SOP)
  - Implement the “ones” of the output
  - Sum all “one” terms → OR results in “one”

- Product of Sums (POS)
  - Implement the “zeros” of the output
  - Multiply “zero” terms → AND results in “zero”

Boolean Equations Example

- You are going to the cafeteria for lunch
  - You will eat lunch (E=1)
  - If it's open (O=1) **and**
  - If they're not serving corndogs (C=0)
- Write a truth table for determining if you will eat lunch (E).

O	*C*	*E*
0	0	
0	1	
1	0	
1	1	

Boolean Equations Example

- You are going to the cafeteria for lunch
  - You will eat lunch (E=1)
  - If it's open (O=1) **and**
  - If they're not serving corndogs (C=0)
- Write a truth table for determining if you will eat lunch (E).

O	*C*	*E*
0	0	0
0	1	0
1	0	1
1	1	0

SOP & POS Form

- SOP – sum-of-products

O	C	E	minterm
0	0		$\overline{O}\ \overline{C}$
0	1		$\overline{O}\ C$
1	0		$O\ \overline{C}$
1	1		$O\ C$

- POS – product-of-sums

O	C	E	maxterm
0	0		$O + C$
0	1		$O + \overline{C}$
1	0		$\overline{O} + C$
1	1		$\overline{O} + \overline{C}$

SOP & POS Form

- SOP – sum-of-products

O	C	E	minterm
0	0	0	$\overline{O}\ \overline{C}$
0	1	0	$\overline{O}\ C$
1	0	1	$O\ \overline{C}$
1	1	0	$O\ C$

- POS – product-of-sums

O	C	E	maxterm
0	0	0	$O + C$
0	1	0	$O + \overline{C}$
1	0	1	$\overline{O} + C$
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SOP & POS Form

- SOP – sum-of-products

O	C	E	minterm
0	0	0	$\overline{O}\ \overline{C}$
0	1	0	$\overline{O}\ C$
1	0	1	$O\ \overline{C}$
1	1	0	$O\ C$

$$E = O\overline{C}$$
$$= \Sigma(m_2)$$

- POS – product-of-sums

O	C	E	maxterm
0	0	0	$O + C$
0	1	0	$O + \overline{C}$
1	0	1	$\overline{O} + C$
1	1	0	$\overline{O} + \overline{C}$

SOP & POS Form

- SOP – sum-of-products

O	*C*	*E*	minterm
0	0	0	$\overline{O}\ \overline{C}$
0	1	0	$\overline{O}\ C$
1	0	1	$O\ \overline{C}$
1	1	0	$O\ C$

$$E = O\overline{C}$$
$$= \Sigma(m_2)$$

- POS – product-of-sums

O	*C*	*E*	maxterm
0	0	0	$O + C$
0	1	0	$O + \overline{C}$
1	0	1	$\overline{O} + C$
1	1	0	$\overline{O} + \overline{C}$

$$E = (O + C)(O + \overline{C})(\overline{O} + \overline{C})$$
$$= \Pi(M0, M1, M3)$$

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Boolean Algebra

- Axioms and theorems to **simplify** Boolean equations
- Like regular algebra, but simpler: variables have only two values (1 or 0)
- **Duality** in axioms and theorems:
  - ANDs and ORs, 0's and 1's interchanged

Boolean Axioms

	Axiom
A1	$B = 0$ if $B \neq 1$
A2	$\overline{0} = 1$
A3	$0 \bullet 0 = 0$
A4	$1 \bullet 1 = 1$
A5	$0 \bullet 1 = 1 \bullet 0 = 0$

Duality

Duality in Boolean axioms and theorems:

- ANDs and ORs, 0's and 1's interchanged

Boolean Axioms

	Axiom
A1	$B = 0$ if $B \neq 1$
A2	$\overline{0} = 1$
A3	$0 \bullet 0 = 0$
A4	$1 \bullet 1 = 1$
A5	$0 \bullet 1 = 1 \bullet 0 = 0$

Boolean Axioms

	Axiom		Dual
A1	$B = 0$ if $B \neq 1$	A1′	$B = 1$ if $B \neq 0$
A2	$\overline{0} = 1$	A2′	$\overline{1} = 0$
A3	$0 \bullet 0 = 0$	A3′	$1 + 1 = 1$
A4	$1 \bullet 1 = 1$	A4′	$0 + 0 = 0$
A5	$0 \bullet 1 = 1 \bullet 0 = 0$	A5′	$1 + 0 = 0 + 1 = 1$

Dual: Exchange: • and +
0 and 1

Boolean Axioms

	Axiom		Dual	Name
A1	$B = 0 \text{ if } B \neq 1$	A1′	$B = 1 \text{ if } B \neq 0$	Binary field
A2	$\overline{0} = 1$	A2′	$\overline{1} = 0$	NOT
A3	$0 \bullet 0 = 0$	A3′	$1 + 1 = 1$	AND/OR
A4	$1 \bullet 1 = 1$	A4′	$0 + 0 = 0$	AND/OR
A5	$0 \bullet 1 = 1 \bullet 0 = 0$	A5′	$1 + 0 = 0 + 1 = 1$	AND/OR

Dual: Exchange: • and +
0 and 1

Basic Boolean Theorems

	Theorem
T1	$B \bullet 1 = B$
T2	$B \bullet 0 = 0$
T3	$B \bullet B = B$
T4	$\overline{\overline{B}} = B$
T5	$B \bullet \overline{B} = 0$

Basic Boolean Theorems: Duals

	Theorem		Dual	Name
T1	$B \bullet 1 = B$	T1′	$B + 0 = B$	Identity
T2	$B \bullet 0 = 0$	T2′	$B + 1 = 1$	Null Element
T3	$B \bullet B = B$	T3′	$B + B = B$	Idempotency
T4		$\bar{\bar{B}} = B$		Involution
T5	$B \bullet \overline{B} = 0$	T5′	$B + \overline{B} = 1$	Complements

Dual: Exchange: • and +
0 and 1

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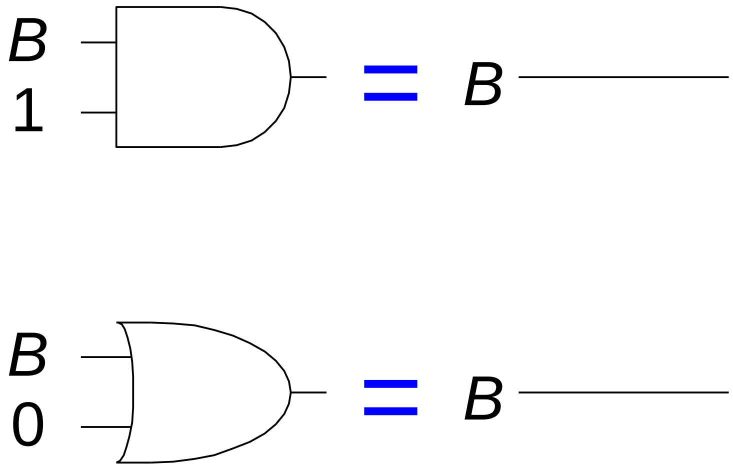
T1: Identity Theorem

- $B \cdot 1 = B$
- $B + 0 = B$

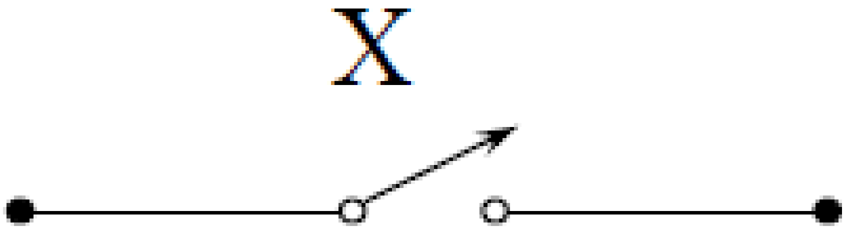
T1: Identity Theorem

- $B \cdot 1 = B$
- $B + 0 = B$



Switching Algebra

- Simplification of digital logic → connecting wires with a on/off switch
- X = 0 (switch open)
- X = 1 (switch closed)



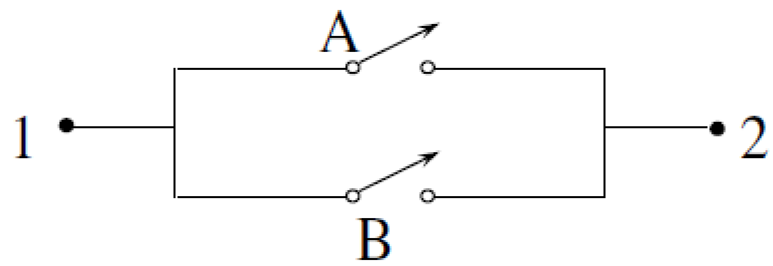
Series Switching Network: AND

- Switching circuit in series performs AND

- 1 is connected to 2 iff A **AND** B are 1

Parallel Switching Network: OR

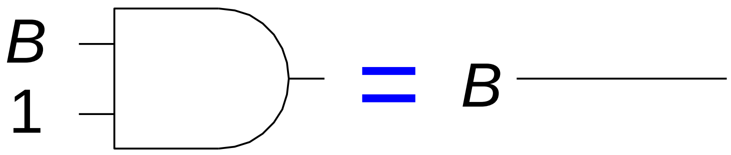
- Switching circuit in parallel performs OR

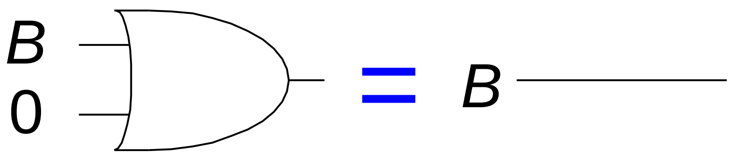


- 1 is connected to 2 if A **OR** B is 1

T1: Identity Theorem

- $B \bullet 1 = B$
- $B + 0 = B$





T2: Null Element Theorem

- $B \cdot 0 = 0$
- $B + 1 = 1$

T2: Null Element Theorem

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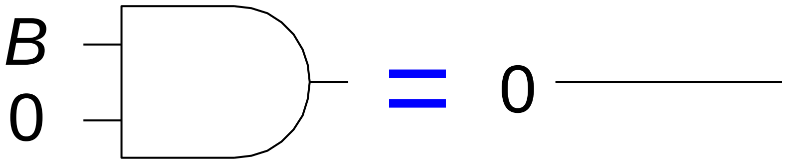
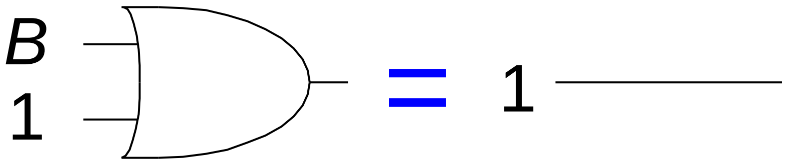


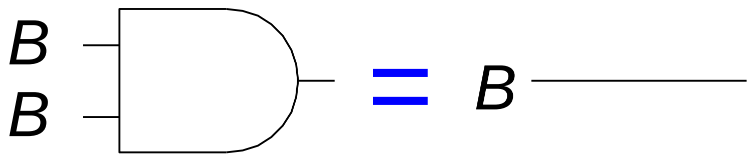


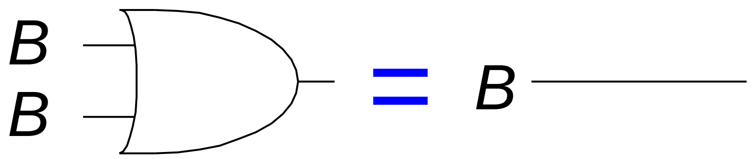
T3: Idempotency Theorem

- $B \cdot B = B$
- $B + B = B$

T3: Idempotency Theorem

- $B \cdot B = B$
- $B + B = B$





T4: Involution Theorem

- $\overline{\overline{B}} = B$

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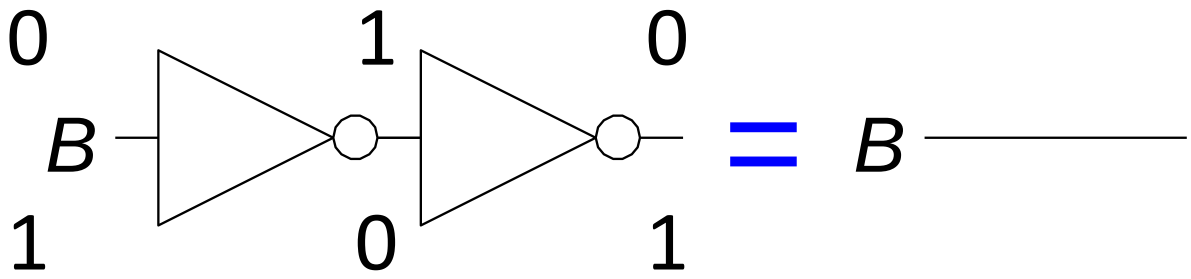
T4: Involution Theorem

- $\overline{\overline{B}} = B$

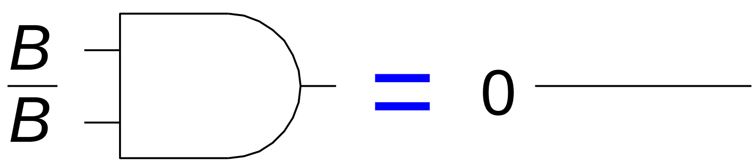


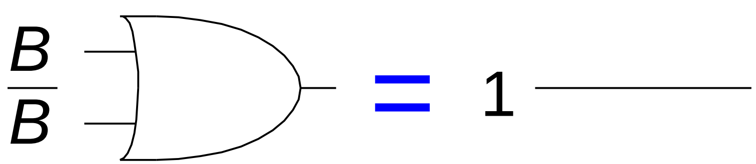
T5: Complements Theorem

- $B \bullet \overline{B} = 0$
- $B + \overline{B} = 1$

T5: Complements Theorem

- $B \cdot \overline{B} = 0$
- $B + \overline{B} = 1$





COMBINATIONAL LOGIC DESIGN

Recap: Basic Boolean Theorems

	Theorem		Dual	Name
T1	$B \bullet 1 = B$	T1′	$B + 0 = B$	Identity
T2	$B \bullet 0 = 0$	T2′	$B + 1 = 1$	Null Element
T3	$B \bullet B = B$	T3′	$B + B = B$	Idempotency
T4		$\bar{\bar{B}} = B$		Involution
T5	$B \bullet \bar{B} = 0$	T5′	$B + \bar{B} = 1$	Complements

Boolean Theorems of Several Vars

Number	Theorem	Name
T6	$B \bullet C = C \bullet B$	Commutativity
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	Associativity
T8	$B \bullet (C + D) = (B \bullet C) + (B \bullet D)$	Distributivity
T9	$B \bullet (B+C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	Combining
T11	$B \bullet C + (\overline{B} \bullet D) + (C \bullet D) =$ $B \bullet C + \overline{B} \bullet D$	Consensus

Boolean Theorems of Several Vars

Number	Theorem	Name
T6	B•C = C•B	Commutativity
T7	(B•C) • D = B • (C • D)	Associativity
T8	B • (C + D) = (B•C) + (B•D)	Distributivity
T9	B• (B+C) = B	Covering
T10	(B•C) + (B•$\overline{C}$) = B	Combining
T11	B•C + ($\overline{B}$•D) + (C•D) = B•C + $\overline{B}$•D	Consensus

How do we prove these are true?

How to Prove Boolean Relation

- **Method 1:** Perfect induction
- **Method 2:** Use other theorems and axioms to simplify the equation
    - Make one side of the equation look like the other

Proof by Perfect Induction

- Also called: proof by exhaustion
- Check every possible input value
- If two expressions produce the same value for every possible input combination, the expressions are equal

Example: Proof by Perfect Induction

Number	Theorem	Name
T6	B•C = C•B	Commutativity

B	*C*	*BC*	*CB*
0	0		
0	1		
1	0		
1	1		

COMBINATIONAL LOGIC DESIGN

Example: Proof by Perfect Induction

Number	Theorem	Name
T6	B•C = C•B	Commutativity

B	*C*	*BC*	*CB*
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

Boolean Theorems of Several Vars

Number	Theorem	Name
T6	$B \bullet C = C \bullet B$	Commutativity
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	Associativity
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T9	$B \bullet (B+C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	Combining
T11	$B \bullet C + (\overline{B} \bullet D) + (C \bullet D) =$ $B \bullet C + \overline{B} \bullet D$	Consensus

T7: Associativity

Number	Theorem	Name
T7	(B•C) • D = B • (C • D)	Associativity

T8: Distributivity

Number	Theorem	Name
T8	B • (C + D) = (B•C) + (B•D)	Distributivity

T9: Covering

Number	Theorem	Name
T9	$B \bullet (B+C) = B$	Covering

T9: Covering

Number	Theorem	Name
T9	B• (B+C) = B	Covering

Prove true by:

- **Method 1:** Perfect induction
- **Method 2:** Using other theorems and axioms

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T9: Covering

Number	Theorem	Name
T9	B• (B+C) = B	Covering

Method 1: Perfect Induction

B	*C*	*(B+C)*	*B(B+C)*
0	0		
0	1		
1	0		
1	1		

T9: Covering

Number	Theorem	Name
T9	B• (B+C) = B	Covering

Method 1: Perfect Induction

B	*C*	*(B+C)*	*B(B+C)*
0	0	0	0
0	1	1	0
1	0	1	1
1	1	1	1

T9: Covering

Number	Theorem	Name
T9	$B \bullet (B+C) = B$	Covering

Method 2: Prove true using other axioms and theorems.

T9: Covering

Number	Theorem	Name
T9	$B\bullet (B+C) = B$	Covering

Method 2: Prove true using other axioms and theorems.

$B\bullet(B+C)$	$= B\bullet B + B\bullet C$	T8: Distributivity
	$= \mathbf{B} + B\bullet C$	T3: Idempotency
	$= B\bullet(1 + C)$	T8: Distributivity
	$= B\bullet(\mathbf{1})$	T2: Null element
	$= B$	T1: Identity

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T10: Combining

Number	Theorem	Name
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	Combining

Prove true using other axioms and theorems:

T10: Combining

Number	Theorem	Name
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	Combining

Prove true using other axioms and theorems:

$$B \bullet C + B \bullet \overline{C} = B \bullet (C + \overline{C}) \quad \text{T8: Distributivity}$$

$$= B \bullet (\mathbf{1}) \quad \text{T5': Complements}$$

$$= B \quad \text{T1: Identity}$$

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T11: Consensus

Number	Theorem	Name
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D) = (B \bullet C) + \overline{B} \bullet D$	Consensus

Prove true using (1) perfect induction or (2) other axioms and theorems.

Recap: Boolean Thms of Several Vars

Number	Theorem	Name
T6	$B \bullet C = C \bullet B$	Commutativity
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	Associativity
T8	$B \bullet (C + D) = (B \bullet C) + (B \bullet D)$	Distributivity
T9	$B \bullet (B+C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	Combining
T11	$B \bullet C + (\overline{B} \bullet D) + (C \bullet D) =$ $B \bullet C + \overline{B} \bullet D$	Consensus

Boolean Thms of Several Vars: Duals

#	Theorem	Dual	Name
T6	B•C = C•B	B+C = C+B	Commutativity
T7	(B•C) • D = B • (C•D)	(B + C) + D = B + (C + D)	Associativity
T8	B • (C + D) = (B•C) + (B•D)	B + (C•D) = (B+C) (B+D)	Distributivity
T9	B • (B+C) = B	B + (B•C) = B	Covering
T10	(B•C) + (B•$\overline{C}$) = B	(B+C) • (B+$\overline{C}$) = B	Combining
T11	(B•C) + (B•$\overline{D}$) + (C•D) = (B•C) + (B•$\overline{D}$)	(B+C) • (B+$\overline{D}$) • (C+D) = (B+C) • (B+$\overline{D}$)	Consensus

Dual: Replace: • with +
0 with 1

Boolean Thms of Several Vars: Duals

#	Theorem	Dual	Name
T6	B•C = C•B	B+C = C+B	Commutativity
T7	(B•C) • D = B • (C•D)	(B + C) + D = B + (C + D)	Associativity
T8	B • (C + D) = (B•C) + (B•D)	B + (C•D) = (B+C) (B+D)	Distributivity
T9	B • (B+C) = B	B + (B•C) = B	Covering
T10	(B•C) + (B•$\overline{C}$) = B	(B+C) • (B+$\overline{C}$) = B	Combining
T11	(B•C) + (B•$\overline{D}$) + (C•D) = (B•C) + (B•$\overline{D}$)	(B+C) • (B+$\overline{D}$) • (C+D) = (B+C) • (B+$\overline{D}$)	Consensus

Dual: Replace: • with +
0 with 1

Warning: T8' differs from traditional algebra: OR (+) distributes over AND (•)

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COMBINATIONAL LOGIC DESIGN

Boolean Thms of Several Vars: Duals

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T6	B•C = C•B	B+C = C+B	Commutativity
T7	(B•C) • D = B • (C•D)	(B + C) + D = B + (C + D)	Associativity
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T10	(B•C) + (B•$\overline{C}$) = B	(B+C) • (B+$\overline{C}$) = B	Combining
T11	(B•C) + (B•$\overline{D}$) + (C•D) = (B•C) + (B•$\overline{D}$)	(B+C) • (B+$\overline{D}$) • (C+D) = (B+C) • (B+$\overline{D}$)	Consensus

Axioms and theorems are useful for *simplifying* equations.

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